Symmetry classification of third-order nonlinear evolution equations

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1 Introduction

In this article we give a Lie-algebraic classification of equations of the form

$$u_t = F(t, x, u, u_x, u_{xx})u_{xxx} + G(t, x, u, u_x, u_{xx}) \tag{1.1}$$

which admit non-trivial Lie point symmetries. Here F and G are arbitrary smooth functions of their arguments, and $F \neq 0$. This paper continues the application of the methods developed and applied in [1, 2, 3, 4, 5, 6], and is a sequel to [4]. Within the class of equations (1.1) are the Korteweg-de Vries (KdV) equation

$$u_t = u_{xxx} + uu_x, \tag{1.2}$$

modified KdV equation

$$u_t = u_{xxx} + u^2u_x, \tag{1.3}$$

the cylindrical KdV equation

$$u_t = u_{xxx} + uu_x - \frac{1}{2t}u, \tag{1.4}$$

the Harry-Dym equation

$$u_t = u^3u_{xxx}, \tag{1.5}$$

as well as the variable coefficient Korteweg-de Vries (vcKdV) equation

$$u_t = f(t, x)u_{xxx} + g(t, x)uu_x \tag{1.6}$$

with $f \neq 0,\ g \neq 0$. The symmetries and integrability properties of equation (1.6) were studied in [7, 8].

In [4], our method of classification was applied to equations of the form

$$u_t = u_{xxx} + G(t, x, u, u_x, u_{xx}) \tag{1.7}$$

which is just equation (1.1) with $F = 1$. Here we allow F to be an arbitrary smooth function (other than the zero function). In particular, in Ref. [4], among others, it was shown that how Eqs. (1.2), (1.3) and (1.4) which is within the class of (1.7) can be recovered from the representative equations of the equivalence classes by changes of point transformations. Eq. (1.5), which is outside of the class (1.7), is known to be integrable and has an infinite hierarchy of generalized symmetries, and therefore admits a Recursion operator [9]. Moreover, it allows a Lax and Hamiltonian formulation. We note that its point symmetry group is isomorphic to the direct sum of an $\mathrm{sl}(2,\mathbb{R})$ algebra with a 2-dimensional nonabelian Lie algebra.

The ideas we exploit are described in [1] and are given more fully in [2]. The mechanism behind our approach is a combination of the usual Lie-algorithm for finding point symmetries of partial differential equations and the equivalence group of the class of equations under study. Here we give a résumé of the steps involved, and we refer the reader to Ref. [2] for details.

The first step is to establish the conditions for a vector field

$$X = a(t,x,u)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$$

to be a symmetry operator for equation (1.1). This gives us the determining equations for the coefficients $a(t,x,u)$, $b(t,x,u)$, $c(t,x,u)$. In general, these equations are not solvable explicitly since the functions $F(t,x,u,u_x,u_{xx})$ and $G(t,x,u,u_x,u_{xx})$ are allowed to be arbitrary.

Our other ingredient is the equivalence group of equation (1.1). This is the group of invertible point transformations which leave invariant the form of the equation. The equivalence group is used to find canonical forms for vector fields which are symmetry operators for the given type of equation. This is the same as linearizing a vector field using diffeomorphisms, but with the difference that the diffeomorphisms allowed belong to a smaller group.

The third part of our calculations involve finding canonical representations for Lie algebras within the class of symmetry operators admitted by the equation. This procedure consists in choosing a canonical representation for one of the operators of the Lie algebra basis and then invoking the commutation relations of the Lie algebra to obtain a form for another of the basis operators. A canonical form for this second operator is found using those transformations of the equivalence group which preserve the (canonical form of the) first operator. This procedure is then continued for all the other basis operators of the Lie algebra. Having done this, we are able to calculate the corresponding functions F and G and this gives us canonical forms for evolution equations of the given type which admit a given Lie algebra as a symmetry algebra. One may think of this method as a systematic way of introducing ansatzes for the forms of the nonlinearities F and G in order to solve the defining equations for the symmetry operator.

Our article is organized as follows: In section 2 we give the determining equations for a symmetry vector field and we calculate the equivalence group of equation (1.1) (in fact, we show that the equivalence transformations for a general evolution equation in $(1+1)$-time-space must be such that the generator of time translations must be a function of the time t only). Although this result is known (see [10], [11]) we give a self-contained proof and improve slightly the statement of the result given in [10], [11]. In sections 3 and 4 we give a detailed discussion of our results for the two semi-simple Lie algebras $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3)$. In particular, we give a detailed account of the classification of the representations of the algebras $\mathrm{so}(3)$ and $\mathrm{sl}(2,\mathbb{R})$ (a sketch of this proof was given in [2]). In section 5 we discuss the classification of those evolution equations (1.1) which admit solvable Lie algebras.

2 Symmetries of evolution equations

2.1 Equivalence Group

The equations we are looking at fall within the class of evolution equations having the general form

$$u_t = F(t, x, u, u_1, u_2, \ldots, u_n) \tag{2.1}$$

with $n \geq 2$ and where u_n stands for $\partial^n u/\partial x^n$, and F is a smooth function of all its arguments with $F_{u_n} \neq 0$. The following result is well known [12]:

Theorem 2.1 *If the smooth vector field $X = a(t,x,u)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$ is a symmetry of equation (2.1) then necessarily $a = a(t)$.*

Proof: The prolongation of X to the n-th jet space gives as symmetry condition

$$\tau_0 = XF + \tau_1 F_{u_1} + \ldots + \tau_n F_{u_n}$$

with equality on the submanifold defined by $u_t = F$ and its differential consequences $D_x^k F = 0,\ k = 1, \ldots, n$. The functions τ_0 and τ_k are defined in the usual way:

$$\tau_0 = D_t\, c - u_t D_t\, a - u_x D_t\, b, \quad \tau_1 = D_x\, c - u_t D_x\, a - u_x D_x\, b,$$
$$\tau_k = D_x \tau_{k-1} - u_{t(k-1)} D_x\, a - u_k D_x\, b,\ k = 2, \ldots, n.$$

Here we have $D_t,\ D_x$ as the usual operators of total differentiation, and $u_{t(k-1)} = D_x^{k-1} u_t$. Calculating the functions τ and evaluating the symmetry condition on the submanifold defined by $u_t = F$ and its differential consequences $D_x^k F = 0,\ k = 1, \ldots, n$, we find that τ_n contains the term

$$u_{t(n-1)} = D_x^{n-1} F = -(D_x\, a) u_{2n-1} F_{u_n} + s_{2n-2}$$

where s_{2n-2} contains derivatives u_k with $1 \leq k \leq 2n-2$. Now, in the symmetry condition, the term $-(D_x\, a) u_{2n-1} F_{u_n}$ is the only one containing u_{2n-1}, and as such it must vanish. Since $F_{u_n} \neq 0$, it follows that $D_x\, a = 0$. But $D_x\, a = a_x + u_x a_u$, and since $a(t,x,u)$ is a function of only (t,x,u), it follows that $a_x = a_u = 0$ and the result follows.

One of the important elements in our analysis is the **equivalence group** of equation (2.1). This is the group of **point transformations**

$$t \to t' = T(t,x,u), \quad x \to x' = X(t,x,u), \quad u \to u' = U(t,x,u)$$

which leave invariant the form of the equation. That is,the equation

$$u'_{t'} = F'(t', x', u', u'_1, \ldots, u'_n).$$

is mapped to the equation (2.1). To this end we have the following result:

Theorem 2.2 *Any invertible point transformation*

$$t' = T(t,x,u), \quad x' = X(t,x,u), \quad u' = U(t,x,u)$$

that maps the evolution equation

$$u'_{t'} = F'(t', x', u', u'_1, u'_2, \ldots, u'_n) \tag{2.2}$$

(for $n \geq 2$) with $\partial F'/\partial u'_n \neq 0$ to an evolution equation

$$u_t = F(t, x, u, u_1, u_2, \ldots, u_n) \tag{2.3}$$

with $\partial F/\partial u_n \neq 0$, is equivalent to an invertible point transformation with

$$t' = T(t), \quad x' = X(t, x, u), \quad u' = U(t, x, u) \tag{2.4}$$

Proof: As before, D_t and D_x denote the operators of total differentiation with respect to t and x. We begin by making the elementary remark that in an evolution equation such as (2.3), the right-hand side contains derivatives with respect to x and no derivatives with respect to t, since it is the evolution parameter. In this case we have $\partial F/\partial u_{0,n} \neq 0$, $\partial F/\partial u_{k,l} = 0$ for $1 \geq k \leq n$, $k + l = n$ where $u_{k,l}$ denotes the derivative $\partial^{k+l}u/\partial t^k \partial x^l$. That is, F is to contain no mixed derivatives of u.

Next we introduce the differential form $DS = D_t S dt + D_x S dx$ for any function $S(t, x, u, u_t, u_x, \ldots, u_{m,0}, u_{m-1,1}, \ldots, u_{0,m})$ on the jet space $J^k(\mathbb{R}^2, \mathbb{R})$. It is easily verified that

$$dS = DS + S_u \theta + \sum_{k,l:1 \leq k+l \leq m} S_{u_{k,l}} \theta_{k,l}$$

where $\theta = du - u_t dt - u_x dx$, $\theta_{k,l} = du_{k,l} - u_{k+1,l} dt - u_{k,l+1} dx$ are the contact forms on the jet space. It is easy to show that $dS = 0$ if and only if $DS = 0$.

Our transformation $(t, x, u) \to (T, X, U)$ preserves the contact condition, that is we have

$$dU - U_0 dT - U_1 dX = \lambda(du - u_t dt - u_x dx)$$

and we define U_0, U_1 as the induced transformations of the derivatives u_t, u_x respectively. For our point transformation this gives

$$DU - U_0 DT - U_1 DX + (U_u - U_0 T_u - U_1 X_u)\theta = \lambda\theta$$

whence $\lambda = U_u - U_0 T_u - U_1 X_u$ and so

$$DU = U_0 DT + U_1 DX.$$

Consequently, $U_1 DT \wedge DX = DT \wedge DU$ defines U_1. This is well-defined since $DT \wedge DX \neq 0$ for an invertible transformation $(t, x, u) \to (T, X, U)$. For if $DT \wedge DX = 0$ we must have that $DT \wedge DU = 0$, by the above, and so $DX = \alpha DT$, $DU = \beta DT$ ($DT \neq 0$ since $dT \neq 0$ for invertibility). Then we find that $dT \wedge dX \wedge dU = (DT + T_u\theta) \wedge (DX + X_u\theta) \wedge (DU + U_u\theta) = 0$, thus contradicting invertibility (the condition for invertibility is $dT \wedge dX \wedge dU \neq 0$).

In the same way we define a sequence of functions U_k with $k \in \mathbb{N}$ by $U_{k+1} DT \wedge DX = DT \wedge DU_k$. These functions give the induced transformations of the spatial

derivatives $u_{0,k}$ for $1 \leq k \leq n$. We note that the highest-order derivatives $u_{k,l}$ in a given U_m are $u_{m,0}, u_{m-1,1}, \ldots, u_{0,m}$. For U_n we require

$$\frac{\partial U_n}{\partial u_{0,n}} \neq 0, \quad \frac{\partial U_n}{\partial u_{n,0}} = \frac{\partial U_n}{\partial u_{n-1,1}} = \cdots = \frac{\partial U_n}{\partial u_{1,n-1}} = 0.$$

This follows from the requirement that F contain only spatial derivatives of u. We have

$$U_n DT \wedge DX = DT \wedge DU_{n-1}. \tag{2.5}$$

Differentiate equation (2.5) with respect to $u_{0,n}$ and the condition on U_n then gives us

$$DT \wedge \frac{\partial U_{n-1}}{\partial u_{0,n-1}} dx = D_t T \frac{\partial U_{n-1}}{\partial u_{0,n-1}} dt \wedge dx \neq 0.$$

Hence, $D_t T \neq 0$ and $\frac{\partial U_{n-1}}{\partial u_{0,n-1}} \neq 0$. The condition $\frac{\partial U_n}{\partial u_{n,0}} = 0$ gives, on differentiating (2.5) with respect to $u_{n,0}$,

$$DT \wedge \frac{\partial U_{n-1}}{\partial u_{n-1,0}} dt = -D_x T \frac{\partial U_{n-1}}{\partial u_{n-1,0}} dt \wedge dx = 0,$$

which implies that either $D_x T = 0$ or $\frac{\partial U_{n-1}}{\partial u_{n-1,0}} = 0$. Assume that $D_x T \neq 0$, so we must have $\frac{\partial U_{n-1}}{\partial u_{n-1,0}} = 0$. Further, the condition $\frac{\partial U_n}{\partial u_{n-k,k}} = 0$ for $1 \leq k \leq n-1$ gives us, on differentiating (2.5) with respect to $u_{n-k,k}$,

$$DT \wedge \left(\frac{\partial U_{n-1}}{\partial u_{n-k-1,k}} dt + \frac{\partial U_{n-1}}{\partial u_{n-k,k-1}} dx \right).$$

It now follows that $\frac{\partial U_{n-1}}{\partial u_{n-k-1,k}} = 0$ if $\frac{\partial U_{n-1}}{\partial u_{n-k,k-1}} = 0$ since we assume $D_x T \neq 0$. This is true for $k = 1$ and hence for $1 \leq k \leq n-1$. That is, U_{n-1} contains no time derivative of u if U_n has the same property, on the assumption that $D_x T \neq 0$. Continuing in this way we come to the conditions

$$\frac{\partial U_2}{\partial u_{2,0}} = \frac{\partial U_2}{\partial u_{1,1}} = 0$$

which give us

$$\frac{\partial U_1}{\partial u_{1,0}} = \frac{\partial U_1}{\partial u_{0,1}} = 0$$

and this is a contradiction since the condition $\frac{\partial U_n}{\partial u_{0,n}} \neq 0$ gives us $\frac{\partial U_1}{\partial u_{0,1}} \neq 0$. Thus we cannot have $D_x T \neq 0$ if $D_x X \neq 0$ and $D_x U \neq 0$. Consequently $D_x T = 0$ and $T = T(t)$ as stated.

We note here that our result is valid for all invertible form-preserving transformations. In [10], [11] the results are for those transformations which are invertible and satisfy $DT \wedge DX \neq 0$ (this is the same as their condition $\delta \neq 0$ where δ is defined by $DT \wedge DX = \delta dt \wedge dx$). Here we have shown this to be a redundant condition.

Corollary 1 *The equivalence group of the evolution equation*

$$u_t = F(t, x, u, u_1, \dots, u_n)u_{n+1} + G(t, x, u, u_1, \dots u_n)$$

where $n \geq 1$ and $F \neq 0$, G are arbitrary smooth functions of their arguments, is the group of point transformations

$$t' = T(t), \quad x' = X(t, x, u), \quad u' = U(t, x, u)$$

where T, X, U are general functions of their arguments.

Proof: From Theorem 2.2, the equivalence group is a subgroup of such transformations. It is a routine calculation to show that all such transformations give an evolution equation of exactly the same form.

2.2 Determining Equations and Symmetry Group

We are interested in classifying the canonical forms of the evolution equation

$$u_t = F(t, x, u, u_1, u_2)u_3 + G(t, x, u, u_1, u_2) \tag{2.6}$$

where F, G are arbitrary functions of their arguments. Its equivalence group is, accordingly, defined by arbitrary functions

$$t' = T(t), \quad x' = X(t, x, u), \quad u' = U(t, x, u)$$

satisfying $\dot{T}(t) \neq 0$ as well as

$$\frac{D(X, U)}{D(x, u)} \neq 0.$$

Lie algebra of the symmetry group of equation (2.6) is realized by vector fields of the form

$$Q = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \phi(t, x, u)\partial_u, \tag{2.7}$$

where τ, ξ, ϕ are arbitrary, real-valued smooth functions defined in some subspace of the space $X \otimes U$. The spaces X and U represent the independent and dependent variables with local coordinates (t, x) and u, respectively.

In order to implement the symmetry algorithm we need to calculate the third order prolongation of the field vector field (2.7) [13] on the jet space $J^3 = X \otimes U^{(3)}$

$$\mathrm{pr}^{(3)}Q = Q + \phi^t\partial_{u_t} + \phi^x\partial_{u_x} + \phi^{xx}\partial_{u_{xx}} + \phi^{xxx}\partial_{u_{xxx}}, \tag{2.8}$$

where

$$\begin{aligned}
\phi^t &= D_t\phi - u_tD_t\tau - u_xD_t\xi, \\
\phi^x &= D_x\phi - u_tD_x\tau - u_xD_x\xi, \\
\phi^{xx} &= D_x\phi^x - u_{xt}D_x\tau - u_{xx}D_x\xi, \\
\phi^{xxx} &= D_x\phi^{xx} - u_{xxt}D_x\tau - u_{xxx}D_x\xi.
\end{aligned}$$

Here D_x and D_t denote the total space and time derivatives. We find the coefficients of the symmetry vector field Q by the requirement that the thrice-prolonged vector field (2.8) annihilates equation (2.6) on its solution manifold

$$\mathrm{pr}^{(3)}Q(\Delta)\Big|_{\Delta=0} = 0, \quad \Delta = u_t - F u_{xxx} - G, \tag{2.9}$$

namely

$$\phi^t - [Q(F) + \phi^x F_{u_x} + \phi^{xx} F_{u_{xx}}]u_{xxx} - \phi^{xxx} F - Q(G) - \phi^x G_{u_x} - \phi^{xx} G_{u_{xx}}\Big|_{u_t = F u_{xxx} + G} = 0. \tag{2.10}$$

Equating coefficients of linearly independent terms of invariance condition (2.10) to zero yields an overdetermined system of linear PDEs (called determining equations). Solving this system we obtain the following assertion.

Proposition 1 *The symmetry group of the nonlinear equation (2.6) for arbitrary (fixed) functions F and G is generated by the vector field*

$$Q = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u \tag{2.11}$$

where the functions a, b and c satisfy the determining equations

$$\begin{aligned} &F\,(-a_t + 3\,u_x\,b_u + 3\,b_x) + [u_x(b_{xx} - 2c_{xu}) + u_x^2(2b_{xu} - c_{uu}) + u_x^3 b_{uu} + \\ &u_{xx}(2b_x - c_u) + 3u_x u_{xx} b_u - c_{xx}]\,F_{u_{xx}} + \\ &[u_x^2\,b_u + u_x(b_x - c_u) - c_x]\,F_{u_x} - c\,F_u - a\,F_t - b\,F_x = 0, \end{aligned} \tag{2.12a}$$

and

$$\begin{aligned} &G\,(-a_t - u_x\,b_u + c_u) - u_x\,b_t + c_t + \\ &F\,[u_x(b_{xxx} - 3c_{xxu}) + 3u_x^2(b_{xxu} - c_{xuu}) + u_x^3(3b_{xuu} - c_{uuu}) + u_x^4 b_{uuu} - \\ &3u_{xx}(c_{xu} - b_{xx}) - 3u_x u_{xx}(c_{uu} - 3b_{xu}) + 6u_x^2 u_{xx} b_{uu} + 3u_{xx}^2 b_u - c_{xxx}] + \\ &[u_x(b_{xx} - 2c_{xu}) + u_x^2(2b_{xu} - c_{uu}) + u_x^3 b_{uu} - u_{xx}(c_u - 2b_x) + 3u_x u_{xx} b_u - c_{xx}]\,G_{u_{xx}} + \\ &[u_x(b_x - c_u) + u_x^2 b_u - c_x]\,G_{u_x} - c\,G_u - a\,G_t - b\,G_x = 0. \end{aligned} \tag{2.12b}$$

Here the dot over a symbol stands for time derivative.

If there are no restrictions on F and G, then (2.12) should be satisfied identically, which is possible only when the symmetry group is a trivial group of identity transformations. The approach we will be taking here is the identification of all specific forms of F, G for which the equation (2.6) admits non-trivial symmetry groups. To achieve this task we use the classical results on classification of low-dimensional Lie algebras obtained mostly in late sixties [14, 15, 16, 17, 18, 19, 20].

The first step towards classification is to make use of the equivalence transformations to find canonical forms for a given vector field. The result is:

Theorem 2.3 *A vector field*

$$Q = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$$

can be transformed by transformations of the form (2.4) into one of the following canonical forms:

$$Q = \partial_t, \qquad Q = \partial_x. \tag{2.13}$$

Proof: Any operator

$$Q = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$$

is transformed by our allowed transformations into

$$Q' = a(t)\dot{T}(t)\partial_{t'} + (aX_t + bX_x + cX_u)\partial_{x'} + (aU_t + bU_x + cU_u)\partial_{u'}.$$

If $a \neq 0$ then either $b = c = 0$ or $b^2 + c^2 \neq 0$. If $b = c = 0$, then we choose $T(t)$ so that $a(t)\dot{T}(t) = 1$ (at least locally). If $b^2 + c^2 \neq 0$ then we have the possibilities $b \neq 0, c = 0$; $b = 0, c \neq 0$; $b \neq 0, c \neq 0$. The case $b \neq 0, c = 0$ is easily disposed of: choose $X = X(x)$, $U = U(u)$. If $b = 0, c \neq 0$ then choose $X = X(u)$, $U = U(x)$. If $b \neq 0, c \neq 0$ then we choose X and U to be each of any two independent integrals of the PDE

$$aY_t + bY_x + cY_u = 0.$$

This the gives the canonical form $Q = \partial_t$ in some coordinate system.

If we now have $a(t) = 0$ then Q is transformed into

$$Q' = (bX_x + cX_u)\partial_{x'} + (bU_x + cU_u)\partial_{u'}.$$

If we now have $c = 0, b \neq 0$ we choose $U = U(u)$ and X is chosen so that $bX_x = 1$. On the other hand, if $b = 0, c \neq 0$ we choose $U = U(x)$ and X is chosen so that $cX_u = 1$. If $b \neq 0, c \neq 0$ then we choose U to be the independent integral of

$$bU_x + cU_u = 0$$

and X is chosen so that

$$bX_x + cX_u = 1.$$

This gives the canonical form $Q = \partial_x$ in some coordinate system.

3 Classification of equations invariant under simple algebras

The lowest order real semi-simple Lie algebras are isomorphic to one of the following three-dimensional algebras:

$$\begin{aligned}
\mathrm{so}(3) &: \; [Q_1, Q_2] = Q_3, \quad [Q_3, Q_1] = Q_2, \quad [Q_2, Q_3] = Q_1; \\
\mathrm{sl}(2, \mathbb{R}) &: \; [Q_1, Q_2] = 2Q_2, \quad [Q_1, Q_3] = -2Q_3, \quad [Q_2, Q_3] = Q_1.
\end{aligned}$$

In our classification of the canonical forms of (2.6) we use canonical forms for realizations of our Lie algebras in terms of vector fields of the form $Q = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$. Here we present a classification of the possible canonical forms for $\mathrm{so}(3)$ and $\mathrm{sl}(2,\mathbb{R})$ using the idea of rank of a realization. This is defined as follows:

Definition 1 *The* **rank** *of a realization $\langle X_1, \ldots, X_n\rangle$ of a Lie algebra is the least integer $r \in \mathbb{N}$ such that all $r+1$-fold exterior products $X_1 \wedge \cdots \wedge X_{r+1} = 0$.*

The exterior product of vector fields is defined as usual (see [22]): they correspond to antisymmetric contravariant tensor fields (also known as multivector fields). The purpose of introducing rank is to have an efficient book-keeping system for the coefficients in the vector fields.

Lemma 1 $\mathfrak{g} = \mathrm{so}(3)$ *has no rank 1 realizations.*

Proof: The commutation relations for $\mathrm{so}(3)$ are

$$[Q_1, Q_2] = Q_3, \quad [Q_2, Q_3] = Q_1, \quad [Q_3, Q_1] = Q_2.$$

If we have a rank 1 realization, then $Q_1 = fX,\ Q_2 = gX,\ Q_3 = hX$ where f, g, h are smooth real functions and X is a smooth vector field. The commutation relations give $fg' - gf' = h,\ gh' - hg' = f,\ hf' - fh' = g$ where $f' = Xf, g' = Xg, h' = Xh$. Then we find that $f^2 + g^2 + h^2 = 0$ and the only real solution is $f = g = h = 0$.

Lemma 2 *If $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R})$ then for the rank 1 realization $Q_1 = fX,\ Q_2 = gX,\ Q_3 = hX$ we have $f^2 + 4gh = 0$.*

Proof: The commutation relations for $\mathrm{sl}(2,\mathbb{R})$ are

$$[Q_1, Q_2] = 2Q_2, \quad [Q_2, Q_3] = Q_1, \quad [Q_3, Q_1] = 2Q_3.$$

With the rank 1 realization given here, the commutation relations give $fg' - gf' = 2g,\ gh' - hg' = f,\ hf' - fh' = 2h$. These give $f^2 + 4gh = 0$.

Lemma 3 *For all rank 2 realizations of* $\mathrm{so}(3)$ *and* $\mathrm{sl}(2,\mathbb{R})$*, all two-fold products are nonzero.*

Proof: From the identity

$$[X, Y \wedge Z]_S = [X, Y] \wedge Z + Y \wedge [X, Z]$$

where $[,]_S$ is the Schouten bracket (see [21]), we find that

$$[Q_1, Q_1 \wedge Q_2]_S = Q_1 \wedge Q_3, \quad [Q_3, Q_1 \wedge Q_3]_S = Q_2 \wedge Q_3, \quad [Q_2, Q_2 \wedge Q_3]_S = Q_2 \wedge Q_1$$

for so(3), and we see that $Q_1 \wedge Q_2 = 0 \iff Q_1 \wedge Q_3 = 0 \iff Q_2 \wedge Q_3 = 0$. Hence, for any realization of rank $r \geq 2$ we must have all two-fold products different from zero.

For $\mathrm{sl}(2, \mathbb{R})$

$$[Q_3, Q_1 \wedge Q_2]_S = -2Q_2 \wedge Q_3, \quad [Q_3, Q_3 \wedge Q_2]_S = Q_1 \wedge Q_3,$$

$$[Q_2, Q_1 \wedge Q_3]_S = -2Q_2 \wedge Q_3, \quad [Q_2, Q_2 \wedge Q_3] = Q_2 \wedge Q_1,$$

and the same result holds.

We first classify Eqs. (2.6) whose symmetry algebras are so(3). We present our results as theorems:

3.1 The case of so(3)

Representations of so(3) by vector fields:

Theorem 3.1 *There exists only one realization of the algebra* so(3) *by vector fields which is an invariance algebra of (2.6) :*

$$\langle \partial_x, \ \tan u \, \sin x \, \partial_x + \cos x \, \partial_u, \ \tan u \, \cos x \, \partial_x - \sin x \, \partial_u \rangle. \tag{3.1}$$

Furthermore, the most general form of the functions F, G allowing for PDE (2.6) to be invariant under the above realization is given by

$$F = \frac{\sec^3 u}{(1+\omega^2)^{\frac{3}{2}}} f(t, \psi),$$

$$G = \left[9\omega\psi \tan u - 3\omega\psi^2 (1+\omega^2)^{\frac{1}{2}} + \frac{\omega(1+2\omega^2)}{(\omega^2+1)^{\frac{3}{2}}} - \frac{\omega(5+6\omega^2)\tan^2 u}{(\omega^2+1)^{\frac{3}{2}}} \right] f(t,\psi) + (\omega^2+1)^{\frac{1}{2}} h(t, \psi),$$

where we have used the notation

$$\omega = u_x \sec u, \quad \psi = \frac{u_{xx} \sec^2 u + (1+2\omega^2) \tan u}{(1+\omega^2)^{\frac{3}{2}}}.$$

Provided the function $\tilde{G}$ is arbitrary, the realization (3.1) is the maximal symmetry algebra of the corresponding equation.

Proof: There are no rank 1 realizations, by lemma 1. If we take the first canonical form of Q_1 as $Q_1 = \partial_t$ and $Q_2 = a(t)\partial_t + X$, $Q_3 = A(t)\partial_t + Y$ where $X = b\partial_x + c\partial_u$, $Y = B\partial_x + C\partial_u$, we find from $[Q_1, Q_2] = Q_3$ that $A = \dot{a}$ and from $[Q_1, Q_3] = -Q_2$ that $a = -\dot{A}$. With this, $[Q_2, Q_3] = Q_1$ then gives $a\dot{A} - A\dot{a} = 1$ which in turn gives $a^2 + \dot{a}^2 = -1$ and this has no real solutions, so we have no realizations with $Q_1 = \partial_t$.

We then take the second canonical form $Q_1 = \partial_x$. $[Q_1, Q_2] = Q_3$ gives $A = 0$ and $[Q_1, Q_3] = -Q_2$ then gives $a = 0$. So $Q_2 = b\partial_x + c\partial_u$, $Q_3 = B\partial_x + C\partial_u$. The commutation relations $[Q_1, Q_2] = Q_3$ and $[Q_1, Q_3] = -Q_2$ give us $B = b_x$, $C = c_x$, $b = -B_x$, $c = -C_x$. Note that we have a rank 2 realization of so(3) and therefore, by Lemma 3, $Q_1 \wedge Q_2 \neq 0$, $Q_1 \wedge Q_3 \neq 0$, $Q_2 \wedge Q_3 \neq 0$ give us $c \neq 0$, $C \neq 0$, $bC - cB \neq 0$. We cannot have $b = 0$ for then $B = 0$ and $bC - cB = 0$, contradicting the requirement of rank.

With $B = b_x$, $C = c_x$, $b = -B_x$, $c = -C_x$ we have $b = b_1(t,u)\cos x + b_2(t,u)\sin x$, $c = c_1(t,u)\cos x + c_2(t,u)\sin x$ and $b_1^2 + b_2^2 \neq 0$, $c_1^2 + c_2^2 \neq 0$ since $b \neq 0$, $c \neq 0$. Thus we may write $b = \alpha(t,u)\cos(x+\phi(t,u))$, $c = \beta(t,u)\cos(x+\theta(t,u))$ for some smooth non-zero functions $\alpha, \beta, \phi, \theta$. Hence we have

$$\begin{aligned} Q_2 &= \alpha(t,u)\cos(x+\phi(t,u))\partial_x + \beta(t,u)\cos(x+\theta(t,u))\partial_u, \\ Q_3 &= -\alpha(t,u)\sin(x+\phi(t,u))\partial_x - \beta(t,u)\sin(x+\theta(t,u))\partial_u. \end{aligned}$$

The equivalence transformations leaving invariant the form of $Q_1 = \partial_x$ are given by $t' = T(t)$, $x' = x + X(t,u)$, $u' = U(t,u)$ with $\dot{T} \neq 0, U_u \neq 0$. Choosing $X = \theta$, $U_u = \beta^{-1}$ we transform Q_2, Q_3 to

$$\begin{aligned} Q_2 &= \alpha(t,u)\cos(x+\gamma(t,u))\partial_x + \cos x\partial_u, \\ Q_3 &= -\alpha(t,u)\sin(x+\gamma(t,u))\partial_x - \sin x\partial_u. \end{aligned}$$

The commutation relation $[Q_2, Q_3] = Q_1$ gives $\cos\gamma = 0$, $\alpha^2 + \alpha_u \sin\gamma = -1$. Hence $\gamma = (2k+1)\pi/2$ with $\sin\gamma = -1$ for otherwise the second equation has no real solutions. We then find

$$Q_2 = \tan(u+\kappa(t))\sin x\partial_x + \cos x\partial_u, \quad Q_3 = \tan(u+\kappa(t))\cos x\partial_x - \sin x\partial_u.$$

The equivalence transformations leaving invariant the form of Q_1, Q_2, Q_3 are given by $t' = T(t), x' = x, u' = u + \lambda(t)$. Applying such a transformation we obtain the stated canonical form for so(3).

We proceed to construct the corresponding invariant equation. The symmetry condition for the given evolution equation gives us four equations:

$$F_u - u_1 \tan u F_{u_1} - (1 + 2u_1^2 \sec^2 u + 2u_2 \tan u)F_{u_2} = 3\tan u F, \tag{3.2}$$

$$-(1+u_1^2\sec^2 u)F_{u_1} + (u_1\tan u - 2u_1^3 \tan u \sec^2 u - 3u_1u_2\sec^2 u)F_{u_2} = 3u_1\sec^2 u F, \tag{3.3}$$

$$\begin{aligned} &G_u - u_1\tan u G_{u_1} - (1 + 2u_1^2\sec^2 u + 2u_2\tan u)G_{u_2} = \\ &(-u_1\tan u + 6u_1^3\sec^3 u\sin u + 9u_1u_2\sec^2 u)F, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &u_1\sec^2 u G - (1+u_1^2\sec^2 u)G_{u_1} + (u_1\tan u - 2u_1^3\sec^3 u\sin u - 3u_1u_2\sec^2 u)G_{u_2} = \\ &[-1 - 3u_1^2\sec^2 u + 2u_1^4\sec^4 u(1+2\sin^2 u) - 3u_2\tan u + 12u_1^2u_2\sec^3 u\sin u + 3u_2^2]. \end{aligned} \tag{3.5}$$

Equation (3.3) gives us the following Lagrangian system:

$$\frac{dt}{0} = \frac{du}{0} = \frac{du_1}{-(1+u_1^2\sec^2 u)} = \frac{du_2}{u_1\tan u - 2u_1^3\tan u\sec^2 u - 3u_1u_2\sec^2 u} = \frac{dF}{3u_1\sec^2 uF}$$

and this gives us the following integrals of motion:

$$t, \quad u, \quad (1+\omega^2)^{3/2}F, \quad \Omega = \frac{u_2+\sin u\cos u(1+2\omega^2)}{(1+\omega^2)^{3/2}},$$

where we have put $\omega = u_1\sec u$. Consequently, the form of F is now

$$F = \frac{1}{(1+\omega^2)^{3/2}}\tilde{F}(t,u,\Omega). \tag{3.6}$$

Now substitute (3.6) into equation (3.2) and we obtain (after making the change of variables $(t,u,u_1,u_2)\to(t,u,\omega,\Omega)$)

$$\tilde{F}_u - 2\tan u\Omega\tilde{F}_\Omega = 3\tan u\tilde{F}.$$

Then this equation gives us the following integrals of motion

$$t, \quad \cos^3 u\,\tilde{F}, \quad \psi = \sec^2 u\,\Omega$$

and from this one sees that our form for $\tilde{F}$ is

$$\tilde{F} = \sec^3 u\, f(t,\psi)$$

which in turn gives us the final form for F

$$F = \frac{\sec^3 u}{(1+\omega^2)^{3/2}}f(t,\psi). \tag{3.7}$$

Our next step is to calculate the corresponding G. We change coordinates

$$(t,u,u_1,u_2)\to(t,u,\omega,\psi)$$

and write for the sake of clarity

$$G(t,u,u_1,u_2) = \tilde{G}(t,u,\omega,\psi).$$

Then equation (3.4) gives us, after some calculation, that

$$\tilde{G}_u = \left[9\omega\psi\sec^2 u - \frac{\tan u\sec^2 u}{(\omega^2+1)^{3/2}}(10\omega+12\omega^3)\right]f(t,\psi).$$

This equation requires careful calculation. It integrates easily to give

$$\tilde{G} = \tilde{g}(t,\omega,\psi) + \left[9\omega\psi\tan u - \frac{\tan^2 u}{(\omega^2+1)^{3/2}}(5\omega+6\omega^3)\right]f(t,\psi). \tag{3.8}$$

Now, note that equation (3.5) gives (after careful calculation)

$$\omega\tilde{G} - (1+\omega^2)\tilde{G}_\omega =$$
$$\begin{bmatrix} -(1+3\omega^2)\sec^2 u + 2\omega^4(\sec^2 u + 2\tan^2 u) - 3\psi(1+\omega^2)^{3/2}\tan u \\ +3(1+2\omega^2)\tan^2 u + 12\psi\omega^2(1+\omega^2)^{3/2}\tan u \\ -12\omega^2(1+2\omega^2)\tan^2 u + 3(1+\omega^2)^3\psi^2 \\ (12\omega^4 + 12\omega + 3)\tan^2 u - 6\psi(1+2\omega^2)(1+\omega^2)^{3/2}\tan u \end{bmatrix} \frac{f(t,\psi)}{(1+\omega^2)^{3/2}}.$$

Also, equation (3.8) gives us (on calculating the left-hand side of the previous equation)

$$\omega\tilde{G} - (1+\omega^2)\tilde{G}_\omega =$$
$$\omega\tilde{g}(t,\omega,\psi) - (1+\omega^2)\tilde{g}_\omega(t,\omega,\psi) + f(t,\psi)\left[-9\psi\tan u + \frac{(5+3\omega^2-6\omega^4)\tan^2 u}{(1+\omega^2)^{3/2}}\right].$$

The right-hand sides of these last two equations must be equal, and on putting them equal and cancelling common terms, one arrives at the following differential equation for $\tilde{g}$

$$\omega\tilde{g} - (1+\omega^2)\tilde{g} = f(t,\psi)\left[3\psi^2(1+\omega^2)^{3/2} + \frac{2\omega^4 - 3\omega^2 - 1}{(1+\omega^2)^{3/2}}\right]$$

or, in a form ready for integration,

$$\frac{\tilde{g}_\omega}{\sqrt{1+\omega^2}}\frac{\omega}{(1+\omega^2)^{3/2}}\tilde{g} = -f(t,\psi)\left[3\psi^2 + \frac{2\omega^4 - 3\omega^2 - 1}{(1+\omega^2)^3}\right]$$

and this equation integrates to give

$$\tilde{g}(t,\omega,\psi) = \sqrt{1+\omega^2}h(t,\psi) + f(t,\psi)\left[\frac{\omega(1+2\omega^2)}{(1+\omega^2)^{3/2}} - 3\omega\sqrt{1+\omega^2}\psi^2\right].$$

Finally, putting all these results together, we arrive at the stated results.

3.2 The case of $\mathrm{sl}(2,\mathbb{R})$.

Representations of $\mathrm{sl}(2,\mathbb{R})$ by vector fields:

Our problem is to represent the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ which is given by the commutation relations

$$[Q_1, Q_2] = 2Q_2, \quad [Q_1, Q_3] = -2Q_3, \quad [Q_2, Q_3] = Q_1$$

by vector fields of the form

$$Q = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u.$$

This is a consequence of Theorem 2.1.

We now turn to finding the representations of our algebra $\langle Q_1, Q_2, Q_3\rangle$. Our results are given in the following theorem:

Theorem 3.2 *Any realization of* $\mathrm{sl}(2,\mathbb{R})$ *as vector fields* $\langle Q_1,\, Q_2,\, Q_3\rangle$, *with the* Q_i *of the form* $Q = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$, *is equivalent under arbitrary invertible smooth transformations of the form*

$$t \to T(t), \quad x \to X(t,x,u), \quad u \to U(t,x,u),$$

to one of the following eight canonical forms:

$$\langle 2t\partial_t, -t^2\partial_t, \partial_t\rangle, \tag{3.9a}$$
$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle, \tag{3.9b}$$
$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x, \partial_t\rangle, \tag{3.9c}$$
$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle, \tag{3.9d}$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle, \tag{3.9e}$$
$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle, \tag{3.9f}$$
$$\langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle, \tag{3.9g}$$
$$\langle 2x\partial_x - u\partial_u, -(u^{-4} + x^2)\partial_x + xu\partial_u, \partial_x\rangle. \tag{3.9h}$$

Proof: We may choose a coordinate system in which Q_3 is in canonical form so that either $Q_3 = \partial_t$ or $Q_3 = \partial_x$. We put $Q_1 = a(t)\partial_t + b\partial_x + c\partial_u,\ Q_2 = A(t)\partial_t + B\partial_x + C\partial_u$.

Rank 1 realizations.For rank 1 realizations we have $Q_1 = fQ_3,\ Q_2 = gQ_3$ and then Lemma 3 gives $f^2 + 4g = 0$. For $Q_3 = \partial_t$ the relation $[Q_1, Q_3] = -2Q_3$ gives us $f_t = 2$, whence $f = 2t + a$ for some constant a since f is a function of t only. The equivalence transformations leaving invariant the form of Q_3 are $t' = t + l,\ x' = X(x,u), u' = U(x,u)$ with $l =$ constant. Under such a transformation Q_1 is transformed to $(2t' + a - 2l)\partial_{t'}$ and we choose l so that $a = 2l$. This gives us the canonical form $Q_1 = 2t\partial_t$ and hence, by Lemma 3, $Q_2 = -t^2\partial_t$. If now $Q_3 = \partial_x$ then a similar argument gives $Q_1 = 2x\partial_x,\ Q_2 = -x^2\partial_x$. The relation $[Q_1, Q_3] = -2Q_3$ gives $f_x = 2$, so $f = 2x + a(t,u)$. Then an equivalence transformation $t' = T(t),\ x' = x + X(t,u), u' = U(t,u)$, leaving the form of Q_3 invariant, maps Q_1 to $(2x' + a - 2X)\partial_{x'}$ and we choose X so that $a = 2X$. We obtain the canonical form $Q_1 = 2x\partial_x$ and $Q_2 = -x^2\partial_x$ by Lemma 2. Thus there are two (canonical) rank 1 realizations of $\mathrm{sl}(2,\mathbb{R})$

$$\langle 2t\partial_t, -t^2\partial_t, \partial_t\rangle, \ \ \langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle.$$

Rank 2 realizations with $Q_3 = \partial_t$. We must have $b^2 + c^2 \neq 0,\ B^2 + C^2 \neq 0$ and $bC - cB = 0$ from the rank 2 conditions of Lemma 3 (the last condition comes from the requirement $Q_1 \wedge Q_2 \wedge Q_3 = 0$ for rank 2 realizations). Using equivalence transformations that preserve the form of Q_3 (these are given by $t' = t, x' = X(x,u), u' = U(x,u)$) we may always choose U so that $bU_x + cU_u = 0$ and $bX_x + cX_u = X$. To see this, let $b \neq 0,\ c = 0$. Then

$$Q_1' = 2t'\partial_{t'} + bX_x\partial_{x'} + bU_x\partial_{u'}$$

and we choose $U = U(u)$ and X so that $bX_x = X$. If $b = 0,\ c \neq 0$ then we choose $U = U(x)$ and X so that $cX_u = X$. If $b \neq 0,\ c \neq 0$ then we choose U to be the

independent integral of $bU_x + cU_u = 0$ and X to be the integral of $bX_x + cX_u = X$. Thus, we obtain the canonical form $Q_1 = 2t\partial_t + x\partial_x$. Since we have rank 2 we must have $C = 0$ because $bC - cB = 0$.The commutation relation $[Q_2, Q_3] = Q_1$ gives $\dot{A}(t) = 2t$, $B_t = -x$ so that $A = -t^2 + k$ and $B = \beta(x, u) - xt$. The relation $[Q_1, Q_2] = 2Q_2$ then gives us $k = 0$, $x\beta_x = 3\beta$ which in turn gives $\beta = 0$ or $\beta = m(u)x^3$ for some smooth function $m(u)$. If $\beta = 0$ then we have $Q_2 = -t^2\partial_t - xt\partial_x$. For $\beta \neq 0$ we have

$$Q_2 = -t^2\partial_t + (m(u)x^3 - xt)\partial_x.$$

Now we seek a canonical form for Q_2 using an equivalence transformation (T, X, U) which preserves the form of Q_1, Q_3. These are (as is easily checked) of the form

$$T(t) = t, \quad X(x, u) = \alpha(u)x, \quad U = \gamma(u).$$

where $\alpha(u) \neq 0$, $\gamma'(u) \neq 0$. Under this transformation we find that Q_2 is transformed to

$$Q_2' = t'\partial_{t'} + \left(\frac{m(u)x'^3}{\alpha(u)^2} - x't'\right)\partial_{x'}.$$

We choose $\alpha(u)$ so that $\alpha(u)^2 = m(u)$ if $m(u) > 0$, and we choose $\alpha(u)$ so that $\alpha(u)^2 = -m(u)$ if $m(u) < 0$ and we obtain the forms

$$Q_2 = -t^2\partial_t + (\pm x^3 - xt)\partial_x$$

(omitting primes). However, we note that $t \to -t$, $x \to x$, $u \to u$ is an element of the equivalence group, and so the above two forms for Q_2 are equivalent. In fact, under this transformation $Q_2^- = -t^2\partial_t + (-x^3 - xt)\partial_x$ is transformed to

$$Q_2' = t^2\partial_t + (-x^3 + xt)\partial_x = -Q_2^+$$

where $Q_2^+ = -t^2\partial_t + (x^3 - xt)\partial_x$. Also Q_3 is transformed to $Q_3' = -Q_3$ and Q_1 is left invariant. This gives us the algebra $\langle Q_1, -Q_2^+, -Q_3\rangle$ which is a Lie-isomorphic to $\langle Q_1, Q_2^+, Q_3\rangle$. That is, $\langle Q_1, Q_2^-, Q_3\rangle$ is equivalent to $\langle Q_1, Q_2^+, Q_3\rangle$, so we are left with one type of representation, namely the representation

$$\langle Q_1, Q_2, Q_3\rangle = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle.$$

Rank 2 realizations with $Q_3 = \partial_x$. Then with $Q_1 = a(t)\partial_t + b\partial_x + c\partial_u$ and $Q_2 = A(t)\partial_t + B\partial_x + C\partial_u$ we note that the relation $[Q_2, Q_3] = Q_1$ implies that $a(t) = 0$. That is,

$$Q_1 = b\partial_x + c\partial_u.$$

Then, writing $Q_2 = A\partial_t + B\partial_x + C\partial_u$, we use the commutation relation $[Q_1, Q_2] = 2Q_2$ to deduce that $A = 0$. Thus in the case of $Q_3 = \partial_x$ we have only rank 2 realizations with $Q_1 = b\partial_x + c\partial_u$, $Q_2 = B\partial_x + C\partial_u$. Rank 2 implies that $c \neq 0, C \neq 0$ and $bC - cB \neq 0$. The relation $[Q_1, Q_3] = -2Q_3$ gives $b_x = 2$, $c_x = 0$, so $b = 2x + \beta(t, u)$ and $c = c(t, u)$. Thus $Q_1 = (2x + \beta(t, u))\partial_x + c(t, u)\partial_u$. Using equivalence transformations

$$T = T(t), \quad X = x + Y(t, u), \quad U = U(t, u)$$

which preserve the form of Q_3 we find that Q_1 is transformed into

$$Q_1' = (2x + \beta + c(t,u)Y_u(t,u))(\partial_{x'} + c(t,u)U_u\partial_{u'}.$$

We may always choose Y, U so that $2x + \beta + cY_u = 2(x+Y)$, $cU_u = -U$ and we obtain $Q_1 = 2x\partial_x - u\partial_u$.

Now consider $Q_2 = B\partial_x + C\partial_u$. The relation $[Q_2, Q_3] = Q_1$ then gives

$$B = -x^2 + \lambda(t,u), \quad C = xu + \mu(t,u).$$

The other relation $[Q_1, Q_2] = 2Q_2$ gives us

$$u\lambda_u = -4\lambda, \quad u\mu_u = -\mu.$$

The solution is

$$\lambda(t,u) = \kappa(t)u^{-4}, \quad \mu(t,u) = \sigma(t)u^{-1}$$

and we have

$$Q_2 = (\kappa(t)u^{-4} - x^2)\partial_x + (xu + \sigma(t)u^{-1})\partial_u.$$

We need to find a canonical form for Q_2 using the equivalence group, but only those elements that preserve the form of both Q_1 and Q_2. It is an elementary calculation that gives the form

$$T = T(t), \quad X = x + p(t)u^{-2}, \quad U = q(t)u, \quad \dot{T} \neq 0, \;\; q(t) \neq 0.$$

The transformed operator Q_2' is then given by

$$Q_2' = (Q_2X)\partial_{x'} + (Q_2U)\partial_{u'}$$

and we require

$$Q_2' = (k(t)u'^{-4} - x'^2)\partial_{x'} + (x'u' + s'(t')u'^{-1})\partial_{u'}$$

which leads to the equations

$$Q_2X = k(t)U^{-4} - X^2, \quad Q_2U = XU + s(t)U^{-1}.$$

These simplify to

$$s(t) = q(t)^2[\sigma(t) - p(t)], \quad k(t) = q(t)^4[p(t)^2 - 2p(t)\sigma(t) + \kappa(t)].$$

We see that we can always choose $p(t) = \sigma(t)$ so that we may always arrange to have $s(t) = 0$. This choice of $p(t)$ gives

$$k(t) = q(t)^4[\kappa(t) - \sigma(t)^2]$$

and we see that we may choose $q(t)$ so that $k(t) = 0$ if $\kappa = \sigma^2$; this gives us the canonical form $Q_2 = -x^2\partial_x + xu\partial_u$ which we have already obtained. We choose $q(t)$

to obtain $k(t) = 1$ if $\kappa > \sigma^2$; and we find that we may choose $q(t)$ so that $k(t) = -1$ if $\kappa < \sigma^2$. All these arguments are local. Thus we find the following forms for Q_2:

$$Q_2 = (u^{-4} - x^2)\partial_x + xu\partial_u, \quad Q_2 = -(u^{-4} + x^2)\partial_x + xu\partial_u.$$

Rank 3 realizations. Here we have $Q_1 \wedge Q_2 \wedge Q_3 \neq 0$. As we have seen, $Q_3 = \partial_x$ gives us only rank 2 realizations, so we only investigate $Q_3 = \partial_t$. As in the rank 2 case we may choose $Q_2 = 2t\partial_t + x\partial_x$. We put $Q_2 = A(t)\partial_t + B\partial_x + C\partial_u$. The rank 3 condition requires $C \neq 0$. Using the commutation relations as before, we find that $A = -t^2$ and $B_t = -x,\ C_t = 0$ so that $B = \beta(x,u) - xt,\ C = C(x,u)$. Further, $x\beta_x = 3\beta,\ xC_x = 2C$, which give $\beta = m(u)x^3$ and $C = g(u)x^2,\ g \neq 0$. We then have $Q_2 = -t^2\partial_t + (m(u)x^3 - xt)\partial_x + \gamma(u)x^2\partial_u$. Those equivalence transformations leaving the form of Q_1 and Q_3 invariant are given by $t' = t, x' = \alpha(u)x, u' = \gamma(u)$ with $\gamma'(u) \neq 0$. Under this transformation Q_2 is mapped to

$$-t^2\partial_t + \left(\frac{\alpha(u)m(u) + g(u)\alpha'(u)}{\alpha(u)^3}x^3 - xt\right)\partial_x + g(u)U_u x^2\partial_u.$$

We choose $\alpha(u),\ U(u)$ so that $g(u)\alpha'(u) + m(u)\alpha(u) = 0$ and $g(u)U_u = 1$ and we find the representation

$$\langle Q_1, Q_2, Q_3\rangle = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle.$$

This gives all the canonical forms listed in the theorem.

Theorem 3.3 *The realizations of the algebras* so(3) *and* sl$(2,\mathbb{R})$, *given in Theorems 3.1, 3.2, exhaust the set of all possible realizations of semi-simple Lie algebras by operators (2.7) which are admitted by PDEs of the form (2.6).*

For a proof of this theorem the reader is referred to [2].

3.3 Classification of sl$(2,\mathbb{R})$ invariant equations

Once we have obtained realizations of the algebra sl$(2,\mathbb{R})$ by vector fields of the form (2.11), the next step is to construct the corresponding invariant equations. The requirement of invariance under sl$(2,\mathbb{R})$ will reduce the number of variables in F and G by three. Hence, invariant Eqs. will depend on arbitrary functions of two variables.

In the following we go through all realizations individually and construct the invariant equations.

1. sl$(2,\mathbb{R}) = \langle 2t\partial_t, -t^2\partial_t, \partial_t\rangle$.

   In this representation, we find that $F = 0$ which is inadmissible.

2. sl$(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle$.

The defining equations are

$$\begin{aligned}
&xF_x - u_1F_{u_1} - 2u_2F_{u_2} = F,\\
&xG_x - u_1G_{u_1} - 2u_2G_{u_2} = -2G,\\
&x^3F_x - 3x^2u_1F_{u_1} - 6(xu_1 + x^2u_2)F_{u_2} = 9x^2F,\\
&x^3G_x - 3x^2u_1G_{u_1} - 6(xu_1 + x^2u_2)G_{u_2} = xu_1 + 6(u_1 + 3xu_2)F.
\end{aligned}$$

They integrate to give

$$F = x^{-3}u_1^{-4}f(u,\sigma), \quad \sigma = \frac{u_2}{u_1^2} + \frac{3}{xu_1},$$

$$G = -\frac{x^{-2}\omega}{4} + x^{-2}\left(\frac{9\sigma}{\omega^2} - \frac{12}{\omega^3}\right)f(u,\sigma) + \frac{1}{x^2\omega}g(u,\sigma), \quad \omega = xu_1$$

3. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle$.

   The defining equations for F and G are:

$$\begin{aligned}
&xF_x - u_1F_{u_1} - 2u_2F_{u_2} = F,\\
&xG_x - u_1G_{u_1} - 2u_2G_{u_2} = -2G,\\
&x^2F_u + 2xF_{u_1} + 2F_{u_2} = 0,\\
&x^2G_u + 2xG_{u_1} + 2G_{u_2} = xu_1.
\end{aligned}$$

   These integrate to give

$$F = xf(\omega,\sigma), \quad G = \frac{u_1^2}{4} + x^{-2}g(\omega,\sigma), \quad \omega = 2u - xu_1, \;\; \sigma = 2u - x^2u_2.$$

4. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle$.

   The defining equations are

$$\begin{aligned}
&u_1F_{u_1} + 2u_2F_{u_2} = -3F,\\
&u_1G_{u_1} + 2u_2G_{u_2} = 0,\\
&u_1F_{u_2} = 0,\\
&u_1G_{u_2} = -3u_2F.
\end{aligned}$$

   They integrate to give

$$F = u_1^{-3}f(t,u), \quad G = -\frac{3}{2}\frac{u_2^2}{u_1^4}f(t,u) + g(t,u).$$

Remark:

The invariant equation has the form

$$u_t = (\frac{u_3}{u_1^3} - \frac{3}{2}\frac{u_2^2}{u_1^4})f(t,u) + g(t,u).$$

In terms of the Schwarzian derivative

$$\{u; x\} = \frac{u_3}{u_1} - \frac{3}{2}\Big(\frac{u_2}{u_1}\Big)^2$$

we have

$$u_t = u_1^{-2}\{u; x\}f(t, u) + g(t, u).$$

Exchanging the roles of the variables x and u, namely by the change of variables

$$\tilde{t} = t, \quad \tilde{x} = u, \quad \tilde{u} = x,$$

and using transformation formulae

$$u_1^{-2}\{u; x\} = -\{\tilde{u}, \tilde{x}\}, \quad \tilde{u}_{\tilde{t}} = -u_t\tilde{u}_{\tilde{x}} = -u_t\tilde{u}_1,$$

(omitting tildes) it is transformed to

$$\frac{u_t}{u_1} = \{u; x\}f(t, x) + g(t, x).$$

The third $\mathrm{sl}(2, \mathbb{R})$ symmetry algebra is further extended by one-dimensional time and space translational generators ∂_t and ∂_x. The corresponding equations are

$$\frac{u_t}{u_1} = \{u; x\}f(x) + g(x), \tag{3.10}$$

$$\frac{u_t}{u_1} = \{u; x\}f(t) + g(t). \tag{3.11}$$

We can use an allowed transformation

$$\tilde{t} = T(t), \quad \tilde{x} = x, \quad \tilde{u} = u, \quad \dot{T} \neq 0$$

to set $f(t) \to 1$.

Further, there is a point transformation

$$\tilde{t} = t, \quad \tilde{x} = x + \int^t g(\xi)\, d\xi, \quad \tilde{u} = u(\tilde{t}, \tilde{x})$$

taking the second equation (3.11) to the Schwarzian KdV equation

$$\frac{u_t}{u_1} = \{u; x\},$$

admitting a six-dimensional Lie symmetry group [4]

$$L = \langle \partial_u, u\partial_u, -u^2\partial_u\rangle \oplus \langle \partial_t, \partial_x, t\partial_t + \frac{x}{3}\partial_x\rangle.$$

5. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_x + \partial_x, -t^2\partial_t - xt\partial_x, \partial_t\rangle$.

   This algebra gives the following defining equations:

$$
\begin{aligned}
&F + 2u_{xx}F_{u_{xx}} + u_xF_{u_x} - xF_x = 0,\\
&2G - 2u_{xx}G_{u_{xx}} - u_xG_{u_x} + xG_x = 0,\\
&t(2G - 2u_xG_{u_{xx}} - u_xG_{u_x} + xG_x) + xu_x = 0,
\end{aligned}
$$

   which leads to the condition

$$
xu_x \equiv 0,
$$

   from which it follows that this realization of $\mathrm{sl}(2,\mathbb{R})$ is not an algebra of invariance for (1.1).

6. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle$.

   The defining equations are

$$
\begin{aligned}
&uF_u + 3u_1F_{u_1} + 5u_2F_{u_2} = -6F,\\
&uG_u + 3u_1G_{u_1} + 5u_2G_{u_2} = G, uF_{u_1} + 4u_1F_{u_2} = 0,\\
&uG_{u_1} + 4u_1G_{u_2} = -9u_2F.
\end{aligned}
$$

   They integrate to give

$$
F = u^{-6}f(t,\sigma), \quad G = ug(t,\sigma) + (12u^{-8}u_1^3 - 9u^{-7}u_1u_2)f(t,\sigma),
$$

   where $\sigma = u^{-5}u_2 - 2u^{-6}u_1^2$.

7. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle$.

   The defining equations are

$$
\begin{aligned}
&uF_u + 3u_1F_{u_1} + 5u_2F_{u_2} = -6F,\\
&uG_u + 3u_1G_{u_1} + 5u_2G_{u_2} = G,\\
&(u + 4u^{-5}u_1^2)F_{u_1} + (4u_1 - 20u^{-6}u_1^3 + 12u^{-5}u_1u_2)F_{u_2} = -12u^{-5}u_1F,\\
&(u + 4u^{-5}u_1^2)G_{u_1} + (4u_1 - 20u^{-6}u_1^3 + 12u^{-5}u_1u_2)G_{u_2} =\\
&(-120u^{-7}u_1^4 - 9u_2 + 120u^{-6} - 12u^{-5}u_2^2)F + 4u^{-5}u_1G.
\end{aligned}
$$

   These integrate to give

$$
F = \frac{u^{-6}}{(1+4\omega^2)^{3/2}}f(t,\sigma),
$$

   where

$$
\sigma = \frac{2\psi - 10\omega^2 - 1}{2(1+4\omega^2)^{3/2}}, \quad \omega = u^{-3}u_1, \quad \psi = u^{-5}u_2
$$

   and

$$
G = ug(t,\sigma)\sqrt{1+4\omega^2} - u\left[12\sigma^2\omega\sqrt{1+4\omega^2} + 21\sigma\omega + \frac{15}{2}\frac{\omega + 6\omega^3}{(1+4\omega^2)^{3/2}}\right]f(t,\sigma).
$$

8. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -(u^{-4}+x^2)\partial_x + xu\partial_u, \partial_x\rangle$.

   Again we obtain,

$$\begin{aligned}
&uF_u + 3u_1F_{u_1} + 5u_2F_{u_2} = -6F,\\
&uG_u + 3u_1G_{u_1} + 5u_2G_{u_2} = G\\
&(u-4u^{-5}u_1^2)F_{u_1} + (4u_1 + 20u^{-6}u_1^3 - 12u^{-5}u_1u_2)F_{u_2} = 12u^{-5}u_1F,\\
&(u-4u^{-5}u_1^2)G_{u_1} + (4u_1 + 20u^{-6}u_1^3 - 12u^{-5}u_1u_2)G_{u_2} =\\
&(120u^{-7}u_1^4 - 9u_2 - 120u^{-6} + 12u^{-5}u_2^2)F - 4u^{-5}u_1G.
\end{aligned}$$

   Define $\omega = u^{-3}u_1,\ \psi = u^{-5}u_2$ and let

$$\sigma_+ = \frac{2\psi - 10\omega^2 - 1}{2(4\omega^2-1)^{3/2}}$$

   if $4\omega^2 > 1$ and

$$\sigma_- = \frac{2\psi - 10\omega^2 - 1}{2(1-4\omega^2)^{3/2}}$$

   if $4\omega^2 < 1$.

   Integrating the defining equations gives us the following results:

$$F = \frac{u^{-6}}{(4\omega^2-1)^{3/2}} f(t,\sigma_+)$$

   and

$$G = ug(t,\sigma_+)\sqrt{4\omega^2-1} - u\left[\frac{15}{2}\frac{(6\omega^3-\omega)}{(4\omega^2-1)^{3/2}} + 21\sigma_+\omega + 12\sigma_+^2\omega\sqrt{4\omega^2-1}\right] f(t,\sigma_+)$$

   if $4\omega^2 > 1$.

$$F = \frac{u^{-6}}{(1-4\omega^2)^{3/2}} f(t,\sigma_-)$$

   and

$$G = ug(t,\sigma_-)\sqrt{1-4\omega^2} - u\left[\frac{15}{2}\frac{(6\omega^3-\omega)}{(1-4\omega^2)^{3/2}} + 21\sigma_-\omega + 12\sigma_-^2\omega\sqrt{1-4\omega^2}\right] f(t,\sigma_-)$$

   if $4\omega^2 < 1$.

4 Admissible abelian Lie algebras

Here we look at admissible abelian Lie algebras $A = \langle e_1, \ldots, e_n\rangle$. We show that there are no such Lie algebras of dimension greater than 4.

4.1 $\dim A = 1$:

There are only two inequivalent such algebras: $A = \langle \partial_t \rangle$ and $A = \langle \partial_x \rangle$.

4.2 $\dim A = 2$:

It follows from the above that there are two cases of A to consider $A = \langle \partial_t, e_2 \rangle$ and $A = \langle \partial_x, e_2 \rangle$.

$\underline{A = \langle \partial_t, e_2 \rangle}$: In this case we must have $e_2 = a\partial_t + b(x,u)\partial_x + c(x,u)\partial_u$ with $a \in \mathbb{R}$, for commutativity. Clearly we can take $a = 0$. The equivalence group of $e_1 = \partial_t$ is $\mathcal{E}(\partial_t) = \{t' = t + k,\ x' = X(x,u),\ u' = U(x,u)\}$. Under an equivalence transformation $e_2 = b(x,u)\partial_x + c(x,u)\partial_u$ is mapped to

$$e_2' = (bX_x + cX_u)\partial_X + (bU_x + cU_u)\partial_U,$$

and we choose X, U so that $bX_x + cX_u = 1,\ bU_x + cU_u = 0$ so that $e_2' = \partial_X$ so we obtain

$$A = \langle \partial_t, \partial_x \rangle$$

in canonical form.

$\underline{A = \langle \partial_x, e_2 \rangle}$: In this case we must have $e_2 = a(t)\partial_t + b(t,u)\partial_x + c(t,u)\partial_u$ for commutativity. The equivalence group of $e_1 = \partial_x$ is $\mathcal{E}(\partial_x) = \{t' = T(t),\ x' = x + Y(t,u),\ u' = U(t,u)\}$ with $\dot{T} \neq 0$. Under an equivalence transformation $e_2 = a(t)\partial_t + b(t,u)\partial_x + c(t,u)\partial_u$ is mapped to

$$e_2' = a(t)\dot{T}\partial_T + (b + aY_t + cY_u)\partial_X + (aU_t + cU_u)\partial_U.$$

If $a \neq 0$ we choose T, Y, U so that $a\dot{T} = 1,\ b + aY_t = aU_t + cU_u = 0$, so that we find

$$A = \langle \partial_t, \partial_x \rangle$$

in canonical form. If $a = 0$ then we have either $c = 0$ or $c \neq 0$. If $c \neq 0$ then choose Y, U so that $b + cY_u = 0,\ cU_u = 1$, thus giving $A = \langle \partial_x, \partial_u \rangle$ in canonical form. If $c = 0$ then we have either $b_u \neq 0$ or $b_t \neq 0,\ b_u = 0$. If $b_u \neq 0$ then we take $b(t,u) = U$, so that $e_2' = U\partial_X$. If, however, $b_u = 0$ then we must have $\dot{b}(t) \neq 0$ for linear independence. In this case we may take $b(t) = T(t)$ and we have $e_2' = T\partial_x$, so that we find the canonical forms

$$A = \langle \partial_x, u\partial_x \rangle, \quad A = \langle \partial_x, t\partial_x \rangle.$$

However, $A = \langle \partial_x, t\partial_x \rangle$ is inadmissible since the equation for G gives $G_x = 0$ from ∂_x as a symmetry, whereas $t\partial_x$ gives $tG_x - u_1 = 0$ and this is a contradiction. Consequently we find three admissible abelian Lie symmetry algebras:

$$A = \langle \partial_t, \partial_x \rangle, \quad A = \langle \partial_x, \partial_u \rangle, \quad A = \langle \partial_x, u\partial_x \rangle.$$

4.3 $\dim A = 3$:

We now turn to three-dimensional abelian Lie algebras. Any such abelian Lie algebra is an abelian extension of one of the two-dimensional canonical forms. We take these in turn.

$A = \langle \partial_t, \partial_x, e_3 \rangle$: In this case we have $e_3 = a\partial_t + b(u)\partial_x + c(u)\partial_u$ for commutativity, with $a \in \mathbb{R}$. We can take $a = 0$ and then $e_3 = b(u)\partial_x + c(u)\partial_u$. The equivalence group of $\langle \partial_t, \partial_x \rangle$ is $\mathcal{E}(\partial_t, \partial_x) = \{t' = t + k,\ x' = x + Y(u),\ u' = U(u)\}$ with $U'(u) \neq 0$ and $k \in \mathbb{R}$. Under an equivalence transformation, e_3 is mapped to $e_3' = (b + xY')\partial_X + cU'\partial_U$. If $c \neq 0$ then choose $Y,\ U$ so that $b + cY' = 0,\ cU' = 1$ and we then have $e_3' = \partial_U$, giving $A = \langle \partial_t, \partial_x, \partial_u \rangle$ in canonical form. If, however, $c = 0$ we must have $b'(u) \neq 0$ for linear independence, and we then take $U(u) = b(u)$ so that $e_3' = U\partial_X$ and we obtain $A = \langle \partial_t, \partial_x, u\partial_x \rangle$ in canonical form.

$A = \langle \partial_x, \partial_u, e_3 \rangle$: In this case we have $e_3 = a(t)\partial_t + b(t)\partial_x + c(t)\partial_u$ for commutativity. The equivalence group of $\langle \partial_x, \partial_u \rangle$ is $\mathcal{E}(\partial_x, \partial_u) = \{t' = T(t),\ x' = x + Y(t),\ u' = u + Z(t)\}$ with $\dot{T}(t) \neq 0$. Under an equivalence transformation, e_3 is mapped to $e_3' = a(t)\dot{T}(t)\partial_T + (b + a\dot{Y})\partial_X + (c + a\dot{Z})\partial_U$. If $a \neq 0$ then choose T, Y, Z so that $a\dot{T} = 1,\ b + a\dot{Y} = c + a\dot{Z} = 0$ and we have $e_3' = \partial_T$ giving $A = \langle \partial_t, \partial_x, \partial_u \rangle$ in canonical form. If $a = 0$ then $e_3' = b(t)\partial_X + c(t)\partial_U$. If $\dot{c} = 0$ then we must have $\dot{b} \neq 0$ for linear independence. In this case take $b(t) = T(t)$ and we have $e_3' = T\partial_X$, giving $A = \langle \partial_x, \partial_u, t\partial_x \rangle$ in canonical form, which is inadmissible: ∂_x and $t\partial_x$ are incompatible as symmetry operators. If $\dot{b} = 0$ then we obtain, by the same reasoning $e_3 = t\partial_u$ in canonical form, and this gives $A = \langle \partial_x, \partial_u, t\partial_u \rangle$ in canonical form, which is inadmissible: ∂_u and $t\partial_u$ are incompatible as symmetry operators. If $\dot{b} \neq 0,\ \dot{c} \neq 0$ we take $b = T$ and so $e_3 = t\partial_x + c(t)\partial_u$ in canonical form, giving $A = \langle \partial_x, \partial_u, t\partial_x + c(t)\partial_u \rangle$ in canonical form, which is inadmissible: ∂_x, ∂_u and $t\partial_x + c(t)\partial_u$ are incompatible as symmetry operators. Consequently, we find only the extension $A = \langle \partial_t, \partial_x, \partial_u \rangle$ in this case.

$A = \langle \partial_x, u\partial_x, e_3 \rangle$: In this case, if $e_3 = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$, then commutativity gives $c = 0,\ b_x = 0$ so we have $e_3 = a(t)\partial_t + b(t,u)\partial_x$. The equivalence group of $\langle \partial_x, u\partial_x \rangle$ is $\mathcal{E}(\partial_x, u\partial_x) = \{t' = T(t),\ x' = x + Y(t,u),\ u' = u\}$. Under an equivalence transformation e_3 is mapped to $e_3' = a\dot{T}\partial_T + (b + aY_t)\partial_X$. If $a \neq 0$ then choose T, Y so that $a\dot{T} = 1,\ b + aY_t = 0$ and we obtain $e_3' = \partial_T$ giving $A = \langle \partial_t, \partial_x, u\partial_x \rangle$ in canonicals form. If $a = 0$ then $e_3' = b(t,u)\partial_X$. Now, either $b_{uu} = 0$ or $b_{uu} \neq 0$. If $b_{uu} = 0$ then $b = \beta_1(t)u + \beta_0(t)$ with either $\dot{\beta}_1 \neq 0$ or $\dot{\beta}_1 = 0,\ \dot{\beta}_0 \neq 0$ for linear independence. If $\dot{\beta}_1 \neq 0$, then take $\beta_1 = T$. If $\dot{\beta}_1 = 0,\ \dot{\beta}_0 \neq 0$, take $\beta_0 = T$ and we obtain the canonical forms $A = \langle \partial_x, u\partial_x, (tu + \beta_0(t))\partial_x \rangle$ and $A = \langle \partial_x, u\partial_x, t\partial_x \rangle$, both of which are inadmissible as symmetry algebras. Thus we are left with $b_{uu} \neq 0$, and we investigate conditions for admissibility of this.

$A = \langle \partial_x, u\partial_x, b(t,u)\partial_x \rangle$ as a symmetry algebra. The operators $\partial_x,\ u\partial_x$ give the system

$$F_x = G_x = 0$$
$$u_1 F_{u_1} + 3u_2 F_{u_2} + 3F = 0$$
$$u_1^2 G_{u_1} + 3u_1 u_2 G_{u_2} + 3u_2^2 F = u_1 G.$$

These integrate to give

$$F = u_1^{-3}\phi(t,u,\omega), \quad G = u_1\psi(t,u,\omega) - 3u_1^2\omega^2\phi(t,u,\omega),$$

with $\omega = u_1^{-3}u_2$.

Now invoke $b(t,u)\partial_x$ as a symmetry operator. This gives the following equation for F:

$$b_u(u_1 F_{u_1} + 3u_2 F_{u_2} + 3F) + u_1^2 b_{uu} F_{u_2} = 0.$$

We know that $u_1 F_{u_1} + 3u_2 F_{u_2} + 3F = 0$ and then $b_{uu} \neq 0$ gives $F_{u_2} = 0$ so that we find $F = u_1^{-3}\phi(t,u)$.

For G we obtain

$$b_u(u_1^2 G_{u_1} + 3u_1 u_2 G_{u_2} + 3u_2^2 F - u_1 G) + b_{uu}(u_1^3 G_{u_2} + 6u_1^2 u_2 b_{uu} F) + u_1^4 b_{uuu} F = u_1 b_t,$$

which yields

$$b_{uu}(u_1^3 G_{u_2} + 6u_1^2 u_2 F) + u_1^4 b_{uuu} F = u_1 b_t,$$

since the coefficient of b_u vanishes.

Substituting the expressions for F and G we find, on changing to new coordinates (t,x,u,u_1,ω), that

$$\phi b_{uuu} + \psi_\omega b_{uu} - b_t = 0.$$

Since ψ is the only term depending on ω we conclude that $\psi = \lambda(t,u)\omega + \kappa(t,u)$ and then

$$\lambda = \frac{b_t}{b_{uu}} - \frac{b_{uuu}}{b_{uu}}\phi$$

so that we have

$$F = u_1^{-3}\phi(t,u), \quad G = \frac{u_2}{u_1^2}\left(\frac{b_t}{b_{uu}} - \frac{b_{uuu}}{b_{uu}}\phi(t,u)\right) + u_1\kappa(t,u) - 3\frac{u_2^2}{u_1^4}\phi(t,u).$$

We therefore find the following canonical forms for three-dimensional abelian Lie algebras:

1. $A = \langle \partial_t, \partial_x, \partial_u \rangle$
2. $A = \langle \partial_t, \partial_x, u\partial_x \rangle$
3. $A = \langle \partial_x, u\partial_x, b(t,u)\partial_x \rangle, \quad b_{uu} \neq 0$

4.4 $\dim A = 4$:

We now come to four-dimensional abelian Lie algebras. Any such abelian Lie algebra is an abelian extension of one of the three-dimensional canonical forms. We take these in turn.

For an abelian extension of the three-dimensional algebra of type 1. we require $e_4 = a\partial_t + b\partial_x + c\partial_u$ with $a, b, c \in \mathbb{R}$, by commutativity, and we see that e_4 is linearly dependent on e_1, e_2, e_3, so we obtain no extension in this case.

For extensions of abelian algebras of type 2. we must have, by commutativity, $e_4 = b(u)\partial_x$. Linear independence requires $b''(u) \neq 0$. Then we proceed as in the classification for $b(t,u)\partial_x$ to find forms for the nonlinearities F and G.

For the algebra of type 3. we have, from $[\partial_x, e_4] = [u\partial_x, e_4] = 0$ that $e_4 = a(t)\partial_t + q(t,u)\partial_x$. The equivalence algebra is $\mathcal{E} = \{t' = T(t),\ x' = x + Y(t,u),\ u' = u\}$. We may have $a \neq 0$ only if $b_t = 0$, and in this case we take T, Y so that $a\dot{T} = 1, b + aY_t = 0$ and then e_4 is mapped to $e_4' = \partial_T$ so that we find the algebra

$$A = \langle \partial_t, \partial_x, u\partial_x, b(u)\partial_x \rangle \ \text{ with } b''(u) \neq 0$$

in canonical form. If $a = 0$ we have $e_4 = q(t,u)\partial_x$. It is clear that we have an extension only if

$$\frac{b_t}{b_{uu}} - \frac{b_{uuu}}{b_{uu}} = \frac{q_t}{q_{uu}} - \frac{q_{uuu}}{q_{uu}}.$$

4.5 $\dim A \geq 5$.

If $b_t \neq 0$ then we have no extensions other than by operators of the form $q(t,u)\partial_x$ in the case of the algebra

$$A = \langle \partial_x, u\partial_x, b(t,u)\partial_x, q(t,u)\partial_x \rangle.$$

For an algebra

$$A = \langle \partial_t, \partial_x, u\partial_x, b(u)\partial_x \rangle \ \text{ with } b''(u) \neq 0$$

we have no extension: the only possible extension operator is $q(u)\partial_x$ with

$$\frac{b_{uuu}}{b_{uu}} = \frac{q_{uuu}}{q_{uu}},$$

which gives us

$$q(u) = C_1 b(u) + C_2 u + C_3, \quad \text{with } C_1, C_2, C_3 \in \mathbb{R},$$

and this is easily seen to give the same algebra as

$$A = \langle \partial_t, \partial_x, u\partial_x, b(u)\partial_x \rangle \ \text{ with } b''(u) \neq 0.$$

5 Four and Five dimensional extensions of semi-simple algebras

We now analyze all possible extensions of semi-simple algebras $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3)$ as a direct or semidirect sum with a solvable ideal (the Levi factor). In the case of a one-dimensional extension, we note that a semi-simple Lie algebra does not have any non-trivial one-dimensional representations, and therefore the only one-dimensional extensions of $\mathrm{sl}(2,\mathbb{R})$ and $\mathrm{so}(3)$ are of the form $\mathrm{sl}(2,\mathbb{R})\oplus A_1$ and $\mathrm{so}(3)\oplus A_1$ for a one-dimensional algebra A_1.

For $\mathrm{so}(3)\oplus A_1 = \mathrm{so}(3)\oplus\langle Q_4\rangle$ we find that we have only one such extension with $Q_4 = a(t)\partial_t$ and we may then use equivalence transformations to put Q_4 into canonical form $Q_4 = \partial_t$.

5.1 Admissible semi-direct extensions of $\mathrm{so}(3,\mathbb{R})$.

5.1.1 Rank 1 extensions.

First we look at representing $< Q_1, Q_2, Q_3 >$ with

$$[Q_1,Q_2]=Q_3,\quad [Q_2,Q_3]=Q_1,\quad [Q_3,Q_1]=Q_2.$$

on a Lie algebra of vector fields of rank 1. Thus we require that there be a vector field X so that

$$[Q_1,X]\wedge X=[Q_2,X]\wedge X=[Q_3,X]\wedge X=0.$$

Now we have

$$Q_1=\partial_x,\quad Q_2=\tan u\sin x\partial_x+\cos x\partial_u,\quad Q_3=\tan u\cos x\partial_x-\sin x\partial_u.$$

We begin by noting that either $[Q_1,X]=0$ or $[Q_1,X]\neq 0$.

$\boxed{[Q_1,X]=0}$. It is straightforward to check that either $[Q_2,X]\neq 0,\ \ [Q_3,X]\neq 0$ or $[Q_2,X]=0,\ \ [Q_3,X]=0$: in fact, if two commutators $[Q_i,X]=0$, then the third also vanishes. Thus, we assume

$$[Q_1,X]=0,\quad [Q_2,X]\neq 0,\quad [Q_3,X]\neq 0.$$

We put

$$X=a(t)\partial_t+b(t,x,u)\partial_x+c(t,x,u)\partial_u.$$

Then $[Q_2,X]\neq 0,\ \ [Q_2,X]\wedge X=0$ gives $[Q_2,X]=g(t,x,u)X$ where $g(t,x,u)\neq 0$ is a smooth function. Since $Q_2=\tan u\sin x\partial_x+\cos x\partial_u$, we conclude that $a(t)=0$ so that $X=b\partial_x+c\partial_u$. Further, $[Q_1,X]=0$ gives $b_x=c_x=0$, so that $b=b(t,u),\ \ c=c(t,u)$. Then we have

$$[Q_2,X]=\left[b_u\cos x-b\tan u\cos x-c(1+\tan^2 u)\sin x\right]\partial_x+\left[c_u\cos x+b\sin x\right]\partial_u.$$

From $[Q_2, X] \wedge X = 0$ we obtain

$$c\left[b_u \cos x - b \tan u \cos x - c(1+\tan^2 u)\sin x\right] = b\left[c_u \cos x + b \sin x\right],$$

from which we obtain

$$[(cb_u - bc_u) - bc] \tan u \cos x = [c^2(1+\tan^2 u) + b^2] \sin x.$$

Since $b_x = c_x = 0$ we conclude that the coefficients of $\cos x$ and $\sin x$ must vanish, so that, in particular, $c^2(1+\tan^2 u) + b^2 = 0$ which gives $b = c = 0$, a contradiction. Hence we cannot have $[Q_1, X] = 0,\ \ [Q_2, X] \neq 0,\ \ [Q_3, X] \neq 0$. So either $[Q_1, X] \neq 0$ or $[Q_i, X] = 0,\ i = 1, 2, 3$. $\boxed{[Q_1, X] \neq 0}$. Since we have $[Q_1, X] \neq 0$ then $[Q_1, X] = f(t, x, u)X$ with $f(t, x, u) \neq 0$, so that if we put $X = a(t)\partial_t + b\partial_x + c\partial_u$ we must have $a(t) = 0$ and therefore $X = b\partial_x + c\partial_u$. As above, if $b = 0$ then $[Q_2, X] = -c(1+\tan^2 u)\sin x \partial_x + Q_2 c \partial_u$ and $[Q_2, X] \wedge X = 0$ gives $c = 0$. If $c = 0$ $[Q_2, X] = [Q_2 b - b \tan u \cos x]\,\partial_x + b \sin x \partial_u$ and $[Q_2, X] \wedge X = 0$ gives $b = 0$. hence $b = 0$ if and only if $c = 0$, so we must have $b \neq 0,\ c \neq 0$ if $X \neq 0$. Assuming this, the condition $[Q_1, X] \wedge X = 0$ gives us $bc_x = cb_x$, which leads to

$$\partial_x \left(\frac{b}{c}\right) = 0.$$

Then we have

$$\begin{aligned}
[Q_2, X] &= \left[Q_2 b - b \tan u \cos x - c(1+\tan^2 u)\sin x\right] \partial_x + \left[Q_2 c + b \sin x\right] \partial_u \\
[Q_3, X] &= \left[Q_3 b + b \tan u \sin x - c(1+\tan^2 u)\cos x\right] \partial_x + \left[Q_3 c + b \cos x\right] \partial_u.
\end{aligned}$$

From these equations, the condition $[Q_2, X] \wedge X = [Q_3, X] \wedge X = 0$ gives us

$$\begin{aligned}
c\left[Q_2 b - b \tan u \cos x - c(1+\tan^2 u)\sin x\right] &= b\left[Q_2 c + b \sin x\right] \\
c\left[Q_3 b + b \tan u \sin x - c(1+\tan^2 u)\cos x\right] &= b\left[Q_3 c + b \cos x\right],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
cQ_2 b - bQ_2 c &= bc \tan u \cos x + c^2(1+\tan^2 u)\sin x + b^2 \sin x \\
cQ_3 b - bQ_3 c &= -bc \tan u \sin x + c^2(1+\tan^2 u)\cos x + b^2 \cos x,
\end{aligned}$$

and dividing by c^2 in both equations, we obtain

$$\begin{aligned}
Q_2\left(\frac{b}{c}\right) &= \frac{b}{c} \tan u \cos x + (1+\tan^2 u)\sin x + \left(\frac{b}{c}\right)^2 \sin x \\
Q_3\left(\frac{b}{c}\right) &= -\frac{b}{c} \tan u \sin x + (1+\tan^2 u)\cos x + \left(\frac{b}{c}\right)^2 \cos x.
\end{aligned}$$

That is, with $w = b/c$,

$$\begin{aligned}
Q_2 w &= w \tan u \cos x + (1+\tan^2 u)\sin x + w^2 \sin x \\
Q_3 w &= -w \tan u \sin x + (1+\tan^2 u)\cos x + w^2 \cos x.
\end{aligned}$$

Now, from the above, we know that $w_x = 0$ so that $Q_2 w = \cos x w_u, \quad Q_3 w = -\sin x w_u$ so we have

$$w_u \cos x = w \tan u \cos x + (1 + \tan^2 u) \sin x + w^2 \sin x$$
$$-w_u \sin x = -w \tan u \sin x + (1 + \tan^2 u) \cos x + w^2 \cos x.$$

multiplying the first equation by $\sin x$ and the second by $\cos x$ and adding the results, we find that

$$w^2 + 1 + \tan^2 u = 0,$$

which is impossible for real $b, \ c$. Hence we conclude that we must have $[Q_1, X] = [Q_2, X] = [Q_3, X] = 0$ if we are to have any extension, and such an extension must be a direct sum extension, which has already been studied.

5.1.2 Extensions of rank > 1.

The admissible abelian algebras of rank greater than one are finite-dimensional of dimension no greater than 4.

$\boxed{\dim A = 4}$. When $\dim A = 4$ and the rank of A is greater than 1, we have rank A equal to 2: $A =< Q_4, Q_5, Q_6, Q_7 >$ and, as we have seen, three of the vector fields are parallel and the other has non-zero wedge product with the other three. From Turkowski's classification, there is only one (up to isomorphism) abelian extension of dimension $\dim A = 4$. In particular

$$[Q_1, Q_4] = -\frac{1}{2}Q_7, \quad [Q_1, Q_5] = \frac{1}{2}Q_6, \quad [Q_1, Q_6] = -\frac{1}{2}Q_5, \quad [Q_1, Q_7] = \frac{1}{2}Q_4$$
$$[Q_2, Q_4] = \frac{1}{2}Q_5, \quad [Q_2, Q_5] = -\frac{1}{2}Q_4, \quad [Q_2, Q_6] = \frac{1}{2}Q_7, \quad [Q_2, Q_7] = -\frac{1}{2}Q_6$$
$$[Q_3, Q_4] = \frac{1}{2}Q_6, \quad [Q_3, Q_5] = -\frac{1}{2}Q_7, \quad [Q_3, Q_6] = -\frac{1}{2}Q_4, \quad [Q_3, Q_7] = \frac{1}{2}Q_5.$$

Suppose that $Q_5 \wedge Q_6 = Q_5 \wedge Q_7 = Q_6 \wedge Q_7 = 0$ and that $Q_4 \wedge Q_5 \neq 0, \quad Q_4 \wedge Q_6 \neq 0, \quad Q_4 \wedge Q_7 \neq 0$. Then

$$[Q_1, Q_5 \wedge Q_7]_S = Q_5 \wedge [Q_1, Q_7] + [Q_1, Q_5] \wedge Q_7 = \frac{1}{2}Q_4 \wedge Q_5,$$

where $[\cdot, \cdot]_S$ is the Schouten bracket. Now if $Q_5 \wedge Q_7 = 0$ we obtain $Q_4 \wedge Q_5 = 0$, contradicting the assumption that Q_4 has non-zero wedge-products with the other vector fields. In the same way, we check that if three of the vector fields have zero wedge-products with each other, then each vector field must have vanishing wedge-product with the other vector fields.

From the above, we see that there is no possible admissible extension of dimension four. Thus we come to $\dim A = 3$. The only abelian algebra is, up to isomorphism, $< Q_4, Q_5, Q_6 >$ with the additional commutation rules

$$[Q_1, Q_4] = 0, \quad [Q_1, Q_5] = Q_6, \quad [Q_1, Q_6] = -Q_5$$
$$[Q_2, Q_4] = -Q_6, \quad [Q_2, Q_5] = 0, \quad [Q_2, Q_6] = Q_4$$
$$[Q_3, Q_4] = Q_5, \quad [Q_3, Q_5] = -Q_4, \quad [Q_3, Q_6] = 0.$$

Now, an admissible abelian Lie algebra of rank at least two is either of rank $A = 2$ or rank $A = 3$. If the rank is two, then two of the vector fields have vanishing wedge-product, say $Q_5 \wedge Q_6 = 0$ and $Q_4 \wedge Q_5 \neq 0,\ Q_4 \wedge Q_6 \neq 0$. Then we have

$$
\begin{aligned}
&[Q_1, Q_4 \wedge Q_5]_S = Q_4 \wedge Q_6,\ \ [Q_1, Q_4 \wedge Q_6]_S = -Q_4 \wedge Q_5,\ \ [Q_1, Q_5 \wedge Q_6]_S = 0,\\
&[Q_2, Q_4 \wedge Q_5]_S = Q_5 \wedge Q_6,\ \ [Q_2, Q_4 \wedge Q_6]_S = 0,\ \ [Q_2, Q_5 \wedge Q_6]_S = -Q_4 \wedge Q_5,\\
&[Q_3, Q_4 \wedge Q_5]_S = 0,\ \ [Q_3, Q_4 \wedge Q_6]_S = Q_5 \wedge Q_6,\ \ [Q_3, Q_5 \wedge Q_6]_S = -Q_4 \wedge Q_6.
\end{aligned}
$$

It is easily seen that if any two wedge-products vanish, then all of them must vanish. Consequently, if A is an admissible abelian Lie algebra semi-direct sum extension, then we must have rank $A = 3$.

Now, we have for $i = 4, 5, 6$

$$Q_i = a_i(t)\partial_t + b_i(t, x, u)\partial_x + c_i(t, x, u)\partial_u.$$

From the commutation relations, we have $a_i(t) = 0,\ \ i = 4, 5, 6$. Thus $< Q_4, Q_5, Q_6 >$ gives us only a representation space of rank two, and consequently we cannot have non-trivial semi-direct sum extensions.

5.2 Admissible semi-direct-sum extensions $\mathrm{sl}(2,\mathbb{R}) \triangleright A$ of $\mathrm{sl}(2,\mathbb{R})$

Here we look at semi-direct sum extensions $\mathrm{sl}(2,\mathbb{R}) \triangleright A$ where A is a solvable Lie algebra.

It is known that every representation of a real semi-simple Lie algebra is completely reducible, so we look at the irreducible, finite-dimensional representations. Since the representation space is a solvable Lie algebra A, then $[A, A]$ is an invariant subspace of A for the representation, from which it follows that all our irreducible representation spaces must be commutative algebras. Since any representation of $\mathrm{sl}(2,\mathbb{R})$ is a direct sum of irreducible representations, it suffices to consider the irreducible representations. Furthermore, the dimension of an admissible commutative algebra is $\dim A \leq 3$. Thus we need only look at commutative algebras A with $\dim A = 1, 2, 3$.

Any finite-dimensional irreducible representation of $\mathrm{sl}(2,\mathbb{R})$ is defined uniquely by a half integer $J = n/2,\ n = 0, 1, 2, \ldots$, so that the dimension of the representation space is $2J + 1 \in \mathbb{N}$. Then if $\mathrm{sl}(2,\mathbb{R}) = \langle e_1, e_2, e_3 \rangle$ and $A = \langle e_4, \ldots, e_{2J+4} \rangle$ we have the following matrix representations of the elements e_1, e_2, e_3 on A:

$$e_1 = \begin{bmatrix} 2J & 0 & \dots & 0 \\ 0 & 2(J-1) & \dots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & \dots & 0 & -2J \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 2J & 0 & \dots & \dots & 0 \\ 0 & 0 & 2J-1 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & 1 & 0 \\ 0 & \dots & \dots & \dots & 0 & 0 \end{bmatrix},$$

$$e_3 = \begin{bmatrix} 0 & \dots & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & \dots & 0 \\ 0 & 2 & 0 & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & \dots & 2J & 0 \end{bmatrix}$$

Admissible semidirect-sum extensions of $\mathrm{sl}(2,\mathbb{R})$:

$\boxed{\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle}$

We have an abelian algebra $\langle Q_1, \dots, Q_{2J+1}\rangle$ which is our irreducible representation space for $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle$. Put $Q_{2J+1} = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$. Using the fact that $[e_3, Q_{2J+1}] = 0$ we see that $\dot{a} = 0,\ b_t = c_t = 0$. Further, the condition $[e_1, Q_{2J+1}] = 2JQ_{2J}$ gives $a = 0$ if $J \neq 0$ and $xb_x = -(2J-1)b,\ xc_x = -2Jc$ from which we deduce that, for $J \neq 0$,

$$Q_{2J+1} = x^{-(2J-1)}b(u)\partial_x + x^{-2J}c(u)\partial_u.$$

Substitute this into the symmetry equation for F we find:

$$\begin{aligned} &3\{u_1x^{-2J+1}b' - (2J-1)x^{-2J}b\}F + \{u_1^2x^{-2J+1}b' - u_1x^{-2J}[(2J-1)b + c'] + 2Jx^{-2J-1}c\}F_{u_1} \\ &+ \{u_1x^{-2J-1}[2J(2J-1)b + 4Jc'] - u_1^2x^{-2J}[2(2J-1)b' + c''] + u_1^3x^{-2J+1}b'' \\ &- u_2[2(2J-1)b + c'] + 3u_1u_2x^{-2J+1}b' - 2J(2J+1)x^{-2J-2}c\}F_{u_2} = x^{-2J+1}bF_x + x^{-2J}cF_u. \end{aligned}$$

Now put $F = x^{-3}u_1^{-4}f(u,\sigma)$ where $\sigma = u_1^{-2}u_2 + 3x^{-1}u_1^{-1}$ and change variables from (t,x,u,u_1,u_2) to (t,x,u,τ,σ) where $\tau = xu_1$ and σ as above. Note that we have

$$\begin{aligned} F_x &= -3\tau^{-4}f - 3\tau^{-5}f_\sigma \\ F_u &= x\tau^{-4}f_u \\ F_{u_1} &= -x^2\tau^{-5}(2\sigma f_\sigma + 4f) + 3x^2\tau^{-6}f_\sigma \\ F_{u_2} &= x^3\tau^{-6}f_\sigma. \end{aligned}$$

Multiply throughout by τ^6 and divide by x^{-2J+1} and we obtain a polynomial of degree three in τ:

$$\begin{aligned} &3\tau^2bf + 3\tau bf_\sigma - \tau^2cf_u + 3f\{\tau^3b' - (2J-1)\tau^2b\} \\ &- (2\sigma f_\sigma + 4f)\{\tau^3b' - \tau^2[(2J-1)b + c'] + 2J\tau c\} + 3\{\tau^2b' - [(2J-1)b + c']\tau + 2Jc\}f_\sigma \\ &+ f_\sigma\{\tau[2J(2J-1)b + 4Jc'] - \tau^2[2(2J-1)b' + c''] + \tau^3b'' - \tau^2\sigma[2(2J-1)b + c'] \\ &3\tau[2(2J-1)b + c'] + 3\tau^3\sigma b' - 3\tau^2b' - 2J(2J+1)c\} = 0. \end{aligned}$$

The coefficient of τ^3 gives us the equation

$$[\sigma b' + b'']f_\sigma = b'f$$

which gives, for $f \neq 0$,

$$\frac{f_\sigma}{f} = \frac{b'(u)}{b''(u) + \sigma b'(u)}$$

which integrates to give

$$f(u, \sigma) = C(u)[\sigma b'(u) + b''(u)]$$

with $C(u) \neq 0$. The coefficient of τ gives us the equation

$$\begin{gathered} 3bf_\sigma - 2Jc(2\sigma f_\sigma + 4f) - 3f_\sigma[(2J-1)b + c'] \\ +f_\sigma[2J(2J-1)b + 4Jc'] + 3f_\sigma[2(2J-1)b + c'] = 0 \end{gathered}$$

and this simplifies to

$$4Jc(\sigma f_\sigma + 2f) = 4Jf_\sigma[(J+1)b + c'],$$

which in turn gives, using $f = C(u)[\sigma b' + b'']$,

$$3J\sigma b'c = J[(J+1)bb' + b'c' - 2cb''].$$

The coefficient of σ must vanish so that we obtain, with $J \neq 0$,

$$b'c = 0, \quad [(J+1)bb' + b'c' - 2cb''] = 0$$

We have $cb' = 0$ and then we have $b' = 0$: if $c \neq 0$ we must have $b' = 0$; on the other hand, if $c = 0$ then we have $(J+1)b(u)b'(u) = 0$ from the second equation, and since $J \geq 0$ we have $bb' = 0$ which gives $(b^2)' = 0$ so that $b =$constant and $b' = 0$. But we know that

$$f(u, \sigma) = C(u)[\sigma b'(u) + b''(u)]$$

so $f = 0$, and hence $F = 0$, when $b' = 0$ and this contradicts the fact that we shall have $F \neq 0$.

$\boxed{\mathrm{sl}(2, \mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle}$ We shall first prove that this algebra is incompatible with any algebra containing a rank 1 realization of an abelian Lie algebra A of dimension $\dim A \geq 4$. This is necessary to eliminate abelian Lie algebras of the form $A = \langle \partial_x, u\partial_x, b(t, u)\partial_x\rangle$. To this end, assume $A = \langle Q_1, Q_2, Q_3, \ldots, Q_{2J+1}\rangle$ with $J \geq 1$, with $Q_i \wedge Q_j = 0$ for all $i, j = 1, \ldots, 2J+1$: this is the condition of being a rank 1 realization. From $[e_1, Q_1] = 2JQ_1$ we find that we have

$$Q_1 = at^{J+1}\partial_t + x^{2J+1}b(\omega, u)\partial_x + x^{2J}c(\omega, u)\partial_u$$

where $\omega = tx^{-2}$. Then $Q_2 = [e_3, Q_1]$ gives us

$$Q_2 = (J+1)at^J\partial_t + x^{2J+1}b_\omega(\omega, u)\partial_x + x^{2J}c_\omega(\omega, u)\partial_u.$$

Then $[Q_1, Q_2] = 0$ yields $a = 0$. So we have

$$Q_1 = x^{2J+1} b(\omega, u)\partial_x + x^{2J} c(\omega, u)\partial_u,$$
$$Q_2 = x^{2J+1} b_\omega(\omega, u)\partial_x + x^{2J} c_\omega(\omega, u)\partial_u.$$

Further, $e_3 = \partial_t$ acts nilpotently and $\mathrm{ad}^{2J+1} e_3(Q_1) = 0$ gives us that $b(\omega, u),\ c(\omega, u)$ are polynomials of degree $2J$ in ω. Further, $[e_2, Q_1] = 0$ gives us

$$\omega^2 b_\omega + b_u = 2J\omega b, \quad \omega^2 c_\omega + c_u = 2J\omega c + 2b,$$

and the solutions are $b(\omega, u) = \omega^{2J} f(\tau), \quad c(\tau, u) = \omega^{2J}(g(\tau) + 2uf(\tau))$ where $\tau = u + \frac{1}{\omega}$. The condition of being rank 1 gives $Q_1 \wedge Q_2 = 0$ and this gives $b(\omega, u)c_\omega(\omega, u) = c(\omega, u)b_\omega(\omega, u)$. We cannot have $c = 0$ for this would imply, using the differential equations above, that $b = 0$ and so $Q_1 = 0$, which is a contradiction. So, $c \neq 0$ and assume $b \neq 0$. Then

$$\frac{c_\omega}{c} = \frac{b_\omega}{b},$$

and this has the solution

$$c(\omega, u) = k(u)b(\omega, u).$$

Substituting this into the differential equations for $b(\omega, u), bc(\omega, u)$ gives

$$k'(u)b(\omega, u) = b(\omega, u),$$

so that $k'(u) = 2$ as $b \neq 0$, and so $k(u) = 2u + l$. Hence we obtain

$$g(\sigma) + 2uf(\sigma) = (2u + l)f(\sigma),$$

whence

$$g(u) + 2uf(u) = (2u + l)f(u).$$

Since $b(\omega, u), c(\omega, u)$ are polynomials of degree $2J$ in ω, it follows that $f(\tau), g(\tau)$ are polynomials of degree $2J$: indeed we know that

$$b(\omega, u) = \omega^{2J} f(\tau), \;\; c(\omega, u) = \omega^{2J}(g(\tau) + 2uf(\tau))$$

with $\tau = u + 1/\omega$, and this means that $b(\omega, u), c(\omega, u)$ are polynomials of degree $2J$ in ω only if $f(\tau),\ g(\tau)$ are polynomials of degree at most $2J$.

From this, it then follows that we have

$$Q_{2J+1} = x^{-2J+1} b(u)\partial_x + x^{-2J} c(u)\partial_u$$

where $b(u), c(u)$ are polynomials with $\deg b(u) \leq 2J,\ \deg c(u) \leq 2J + 1$ and

$$c(u) = (2u + l)b(u).$$

In fact, since $f(\sigma)$ is a polynomial of degree no more than $2J$, it follows from a Taylor expansion in $1/\omega$ about u gives $b(u) = f(u)$ and $c(u) = g(u) + 2uf(u)$, with $f(u),\ g(u)$ having degree at most $2J$.

We calculate with $Q_{2J} = [e_2, Q_{2J+1}]$ that

$$Q_{2J} = 2JtQ_{2J+1} + x^{-2J+3}b'(u)\partial_x + x^{-2J+2}[c'(u) - 2b(u)]\partial_u,$$

and the condition $[Q_{2J}, Q_{2J+1}] = 0$ gives us the system of equations

$$\begin{aligned} 4bb' + cb'' &= c'b', \\ 4(J-1)b^2 + 2(J-1)b'c - 2(J-2)bc' + cc'' &= (c')^2. \end{aligned}$$

Note that if $b = 0$ we obtain $cc'' = (c')^2$, and if $c' \neq 0$ we find that $c(u) = ae^{ku}$ which is not a polynomial unless $a = 0$. If $b = 0,\ c' = 0$ we have

$$Q_{2J+1} = x^{-2J}\partial_u.$$

So we now assume $b \neq 0$. Using $c(u) = (2u + l)b(u)$ in the first equation of this system yields

$$2bb' + (2u + l)bb'' = (2u + l)(b')^2.$$

If $b' \neq 0$ this gives the equation

$$\frac{b'}{b} = \frac{b''}{b'} + \frac{2}{2u + l},$$

and this integrates to give

$$b(u) = 2m(2u + l)b'(u)$$

for $m \neq 0$ and then we find that

$$b(u) = K(2u + l)^m, \quad c(u) = K(2u + l)^{m+1}.$$

Clearly, m must be a positive integer for $b(u)$ to be a polynomial, and since $m \neq 0$ we have $m \geq 1$. We may, without loss of generality, put K=1.

If $b'(u) = 0$ then $b(u) =$ const. so we may take $b(u) = 1$ and we then obtain $c(u) = 2u + l$. Thus we have

$$b(u) = (2u + l)^m, \quad c(u) = (2u + l)^{m+1}.$$

with $0 \leq m \leq 2J$ and m an integer.

The equivalence transformations of our realization of sl$(2, \mathbb{R})$ have the form $t' = t,\ x' = ax,\ u' = a^2u + k$ with $a \neq 0$. Choosing $a = 1,\ 2k = -l$ we find that $b(u) = 2^m u^m,\ c(u) = 2^{m+1}u^{m+1}$, and we may, without loss of generality, divide out the factor 2^m to obtain

$$b(u) = u^m, \quad c(u) = 2u^{m+1}$$

in canonical form. Thus we have two canonical forms for Q_{2J+1}

$$\begin{aligned} Q_{2J+1} &= x^{-2J}\partial_u \\ Q_{2J+1} &= x^{-2J+1}u^m\partial_x + 2x^{-2J}u^{m+1}\partial_u. \end{aligned}$$

We go through each of these cases.

If $Q_{2J+1} = x^{-2J}\partial_u$ then $Q_{2J} = [e_2, Q_{2J+1}] = 2JtQ_{2J+1}$. Applying Q_{2J+1} to the symmetry equation for G gives us an equation $E(Q_{2J+1}) = 0$ and then doing the same for Q_{2J} we find $2J(tE(Q_{2J+1}) + x^{-2J}) = 0$ which is a contradiction since $J > 0$. Thus we do not have an admissible symmetry in this case.

If $Q_{2J+1} = x^{-2J+1}u^m\partial_x + 2x^{-2J}u^{m+1}\partial_u$ then for $m = 0$ we find $Q_{2J} = 2JtQ_{2J+1}$ and the same argument as above leads to an inadmissible operator. If $m = 1$ we obtain

$$\begin{aligned} Q_{2J} &= 2JtQ_{2J+1} + X \\ Q_{2J-1} &= J(2J-1)t^2Q_{2J+1} + (2J-1)tX, \end{aligned}$$

with $X = x^{-2J+3}\partial_x + 2x^{-2J+2}u\partial_u$ and the above argument leads to an inadmissible operator.

So we are left with $m \geq 2$. The form of Q_{2J+1} is

$$Q = x^{p+1}u^q\partial_x + 2x^pu^{q+1}\partial_u.$$

If such an operator is a symmetry, then the equation for $F = xf(\omega, \sigma)$, with $\omega = 2u - xu_1$, $\sigma = 2u - x^2u_2$, gives (after a tedious calculation) a polynomial equation in u (after changing to new coordinates (x, u, ω, σ). There are three powers of u which enter: u^{q+1}, u^q, u^{q-1}. Then we obtain from the coefficient of u^{q+1}:

$$q(q-1)f_\sigma = 0,$$

from which we have $f_\sigma = 0$. Then for $J \geq 1$ the other equations are

$$(p + 2q - 2)\omega f_\omega + (3p + 6q + 2)f = 0, \quad q(\omega f_\omega + 3f) = 0.$$

Combining these, we find that $f = 0$, so that $F = 0$, which is a contradiction. So in this case we have no irreducible representation of $\mathrm{sl}(2, \mathbb{R})$ on an abelian Lie algebra of rank 1.

We now turn to representing $\mathrm{sl}(2, \mathbb{R})$ on four-dimensional abelian Lie algebras of rank 2 with a distinguished operator: that is, Lie algebras $A = \langle Q_1, Q_2, Q_3, Q_4\rangle$ where one of the operators (the distinguished operator) has nonzero wedge product with the other three, whereas the other three operators are parallel to each other (and so they constitute a three-dimensional rank 1 abelian Lie algebra). Then the general structure of the operators is

$$\begin{aligned} Q_4 &= x^{-2}b(u)\partial_x + x^{-3}c(u)\partial_u, \\ Q_3 &= 3tQ_4 + X, \\ Q_2 &= 3t^2Q_4 + 2tX + \frac{1}{2}Y, \\ Q_1 &= t^3Q_4 + t^2X + \frac{t}{2}Y. \end{aligned}$$

Here $X,\ Y \neq 0$. If $Q_4 \wedge Q_3 \neq 0$ then all other such products are different from zero, so that we cannot have the type of algebra required. So let $Q_4 \wedge Q_3 = 0$. Then Q_4 is not

a distinguished operator, and so there is one other operator parallel to Q_4. However, if Q_4 is parallel to Q_3 and Q_1 or Q_2, then all wedge products are zero, and we have a rank 1 abelian Lie algebra. Consequently, we do not have $\mathrm{sl}(2,\mathbb{R})$ realized on this type of rank 2 abelian Lie algebra.

Having proved this, we conclude that we need only look at abelian algebras A with $\dim A \leq 3$, since the only admissible abelian algebras with $\dim A > 3$ are those that contain rank 1 realizations of abelian algebras of dimension at least 3 (that is $J \geq 1$).

For $J = 1$ we have $A = \langle Q_1, Q_2, Q_3\rangle$. Commutativity: $[Q_1, Q_2] = [Q_1, Q_3] = [Q_2, Q_3] = 0$ gives the system

$$4bb' + cb'' = 0, \quad 2bc' + cc'' = (c')^2,$$

and it follows from the first equation that $b(u)$ must be a polynomial of degree 1: the term bb' has degree 3 whereas cb'' has degree 2 so the coefficient of u^2 in $b(u)$ must vanish. But then $b'' = 0$ so that $bb' = 0$ and this gives us $b =$ constant. It then follows from the second equation that $c(u)$ is affine linear in u and we have $c(u) = \alpha u + \beta$ with $\alpha = 0$ or $\alpha = 2b$ (which follows from the second equation). If $\alpha = 0$ then we have $b \neq 0$ and this gives $Q_3 = bx^{-1}\partial_x + cx^{-2}$ with $b, c \in \mathbb{R}$. If $c = 0$ then $b \neq 0$ and we may take $Q_3 = x^{-1}\partial_x$. If $c \neq 0$ we may assume $c = 1$ and we find $Q_3 = bx^{-1}\partial_x + x^{-2}\partial_u$. If $\alpha \neq 0$ then we have $\alpha = 2b$ and $b \neq 0$. The equivalence algebra of our $\mathrm{sl}(2,\mathbb{R})$ is $t' = t,\ x' = ax,\ u' = a^2u + k$ with $a \neq 0$, so we may use an equivalence transformation to transform $c(u)$ to $c = \alpha u$ since $2bu + \beta = 2bu' + \beta - 2kb$, and we may choose k so that $\beta - 2kb = 0$. Thus we have the three canonical forms

$$Q_3 = x^{-1}\partial_x + 2x^{-2}u\partial_u, \quad Q_3 = x^{-1}\partial_x, \quad Q_3 = bx^{-1}\partial_x + x^{-2}\partial_x.$$

If $Q_3 = x^{-1}\partial_x$ then $Q_2 = 2tQ_3 - 2b\partial_u$ and $Q_1 = t^2Q_3 - 2tb\partial_u$. Substituting these into the equation for G we obtain the equations

$$bG_u = 2x^{-1}u_1, \quad 2tbG_u = 2tx^{-1}u_1 - 2b,$$

which give $x^{-1}u_1 = 0$, which is a contradiction.

If $Q_3 = bx^{-1}\partial_x + x^{-2}\partial_x$ then $Q_2 = 2tQ_3 - 2b\partial_u,\ Q_1 = t^2Q_3 - 2tb\partial_u$ and, as above, we find that on substituting these into the equation for G we obtain the contradiction $u_1 = 0$.

If $Q_3 = x^{-1}\partial_x + 2x^{-2}u\partial_u$ then we find that $Q_2 = 2tQ_3$ and $Q_1 = t^2Q_3$. On using Q_3 and Q_2 as symmetries, the equations for G then give us $2ux^{-2} - x^{-1}u_1 = 0$ which is a contradiction. Thus there is no realization in this case.

For $J = \frac{1}{2}$ we have $A = \langle Q_1, Q_2\rangle$, and the same reasoning as above gives us $Q_2 = b(u)\partial_x + x^{-1}c(u)\partial_u$, and we find that b, c are polynomials of degree 1 satisfying

$$bb' = 0, \quad 2b^2 + (c')^2 = 3bc'.$$

So $b =$ constant. If $c = \alpha u + \beta$ we then find from the second equation that $2b^2 + \alpha^2 = 3b\alpha$, from which we find $\alpha = b$ or $\alpha = 2b$. If $b = 0$ then $c =$ constant and we have

$Q_2 = x^{-1}\partial_u$. If $b \neq 0$ we may use an equivalence transformation if necessary, as above, and we may assume that $\beta = 0$ and then we find three canonical forms for Q_2:

$$Q_2 = x^{-1}\partial_u, \quad Q_2 = \partial_x + x^{-1}u\partial_u, \quad Q_2 = \partial_x + 2x^{-1}u\partial_u.$$

If $Q_2 = x^{-1}\partial_u$ then $Q_1 = tQ_2$ and substituting these into the equation for G we find $x^{-1} = 0$ which is a contradiction.

If $Q_2 = \partial_x + 2x^{-1}u\partial_u$ then $Q_1 = tQ_2$ and substituting these into the equation for G gives us $2x^{-1}u - u_1 = 0$ which is a contradiction.

If $Q_2 = \partial_x + x^{-1}u\partial_u$ then $Q_1 = tQ_2 - x\partial_u$ and these two symmetries give us a non-trivial equation:

$$\begin{aligned} &F = Kx(2u - 2xu_1 + x^2u_2)^{-1/3}, \quad K \neq 0, \\ &G = x^{-2}\left(xuu_1 - u^2 + L(2u - 2xu_1 + x^2u_2) - 3K(2u - 2xu_1 + x^2u_2)^{2/3}\right). \end{aligned}$$

$\boxed{\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle}$. With $Q_{2J+1} = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$ we have from $[e_3, Q_{2J+1}] = 0$ that $b_x = c_x = 0$ and then from $[e_1, Q_{2J+1}] = -2JQ_{2J+1}$ we find that $Ja = 0$, $(J-1)b = 0$, $Jc = 0$. Hence, $a = c = 0$ if $J \neq 0$ and $b = 0$ if $J \neq 1$. Thus, we have no extension if $J \neq 0, 1$. The case $J = 0$ corresponds to a direct-sum extension, so we consider $J = 1$. Then there are three operators Q_1, Q_2, Q_3 and from the above $Q_3 = b(t,u)\partial_x$. We then have $Q_2 = [e_2, Q_3] = [-x^2\partial_x, b(t,u)\partial_x] = 2xb(t,u)\partial_x$. From this we find $[Q_2, Q_3] = -2b^2\partial_x$ so that $b = 0$ in order that $[Q_2, Q_3] = 0$ and we have no realization in this case.

$\boxed{\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle}$. As above, putting $Q_{2J+1} = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$ we have from $[e_3, Q_{2J+1}] = 0$ that $b_x = c_x = 0$. Then $[e_1, Q_{2J+1}] = -2JQ_{2J+1}$ gives $Ja = 0$, $ub_u = 2(J-1)b$, $uc_u = (2J+1)c$. We consider $J \neq 0$, which gives $a = 0$ and the equations for b, c then give us

$$Q_{2J+1} = u^{2(J-1)}b(t)\partial_x + u^{2J+1}c(t)\partial_u.$$

We then have $Q_{2J} = [e_2, Q_{2J+1}] = 2JxQ_{2J+1} - u^{2J-1}b(t,u)\partial_u$. The condition $[Q_{2J}, Q_{2J+1}] = 0$ gives us $(J+1)bc = 0$, $(2J-1)b^2 = 0$. Since $J \geq 0$ we have $bc = 0$, so either $b = 0$ or $c = 0$, and we see that $b = 0$ if $J \neq 1/2$. Thus, we have $b \neq 0$ only if $J = 1/2$ and in this case we have $Q_{2J+1} = Q_2 = u^{-1}b(t)\partial_x$ and $Q_1 = [e_2, Q_2] = xu^{-1}b(t)\partial_u$. Then $[Q_1, Q_2] = 0$ gives $-2b^2\partial_x = 0$ so that $b = 0$ and we have no realization in this case. For $J \neq 1/2$ we consider $c \neq 0$ so that $Q_{2J+1} = u^{2J+1}c(t)\partial_u$. The equivalence group $\mathcal{E}$ of this realization of $\mathrm{sl}(2,\mathbb{R})$ is $\mathcal{E} = \{t' = T(t),\ x' = x,\ u' = \gamma(t)u\}$ with $\dot{T} \neq 0$, $\gamma(t) \neq 0$. Using an equivalence transformation we find that Q_{2J+1} is mapped to $Q'_{2J+1} = U^{2J+1}c(t)\gamma^{-2J}\partial_U$, and we may choose γ so that $c(t)\gamma^{-2J} = \pm 1$, which gives the canonical form $Q_{2J+1} = \pm u^{2J+1}\partial_u$. Taking $Q_{2J+1} = u^{2J+1}\partial_u$ in the symmetry equation for F we find

$$u^2F_u + (2J+1)uu_1F_{u_1} + \{2J(2J+1)u_1^2 + (2J+1)uu_2\}F_{u_2} = 0.$$

Then taking $F = u^{-6}f(t,\sigma)$ with $\sigma = u^{-5}u_2 - 2u_1^2u^{-6}$ and putting $\tau = u_1u^{-3}$, we obtain the above equation becomes

$$J(2J-1)\tau^2 f_\sigma + (J-2)\sigma f_\sigma - 3f = 0.$$

The coefficient of τ^2 must vanish since $f(t,\sigma)$ is independent of τ, so we find that $J(2J-1)f_\sigma = 0$ and then we must also have $(J-2)\sigma f_\sigma - 3f = 0$. Since we have $J \neq 0, 1/2$ we find that $f_\sigma = 0$, and then the second equation gives $f = 0$, so that we then have $F = 0$, which is not allowed. Thus we have no irreducible semi-direct sum extension except for $J = 0$ (which is a direct-sum extension).

$\boxed{\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x \rangle}$. In this case we have, as above,

$$Q_{2J+1} = u^{2(J-1)}b(t)\partial_x + u^{2J+1}c(t)\partial_u$$

for $J \neq 0$. Then $[e_2, Q_{2J+1}] = Q_{2J}$ yields

$$Q_{2J} = 2JxQ_{2J+1} + 4u^{2(J-2)}c(t)\partial_x - u^{2J-1}b(t)\partial_u.$$

Requiring $[Q_{2J+1}, Q_{2J}] = 0$ yields the equations

$$\begin{aligned}(2J-1)b^2 + 4(J-2)c^2 =& 0\\ (J+1)bc =& 0.\end{aligned}$$

Since $J+1 \geq 1$, the second equation gives us $b = 0$ or $c = 0$. The first equation gives us $b = c = 0$ if $J \neq 1/2, 2$. Thus we have two cases to consider: $J = 1/2,\ c = 0$ and $J = 2,\ b = 0$. For $J = 1/2,\ c = 0$ we have a two-dimensional representation with $Q_2 = u^{-1}b(t)\partial_x$ and $Q_1 = xu^{-1}b(t)\partial_x - b(t)\partial_u$. Then we must have $[Q_1, e_2] = 0$ and this gives us

$$[Q_1, e_2] = 3b(t)u^{-5}\partial_x = 0$$

which gives $b = 0$ and therefore we have no representation for $J = 1/2$.

For $J = 2$ we have $b = 0$ and we then have $Q_5 = c(t)u^5\partial_u$. Then we find, after a tedious calculation, that $(\mathrm{ad}e_2)^5 Q_5 = 0$, as required by the representation (both e_3 and e_2 act nilpotently and have nilpotent order $2J+1 = 5$), only if $c(t) = 0$ and thus there is no realization in this case. So this realization of $\mathrm{sl}(2,\mathbb{R})$ has an irreducible extension only for $J = 0$.

$\boxed{\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -(u^{-4} + x^2)\partial_x + xu\partial_u, \partial_x \rangle}$. In this case we have, as above,

$$Q_{2J+1} = u^{2(J-1)}b(t)\partial_x + u^{2J+1}c(t)\partial_u$$

for $J \neq 0$. Then $[e_2, Q_{2J+1}] = Q_{2J}$ yields

$$Q_{2J} = 2JxQ_{2J+1} - 4u^{2(J-2)}c(t)\partial_x - u^{2J-1}b(t)\partial_u.$$

Requiring $[Q_{2J+1}, Q_{2J}] = 0$ yields the equations

$$\begin{aligned}(2J-1)b^2 - 4(J-2)c^2 =& 0\\ (J+1)bc =& 0.\end{aligned}$$

Since $J+1 \geq 1$, the second equation gives us $b = 0$ or $c = 0$. The first equation gives us $b = c = 0$ if $J \neq 1/2, 2$. Thus we have two cases to consider: $J = 1/2,\ c = 0$ and $J = 2,\ b = 0$. For $J = 1/2,\ c = 0$ we have a two-dimensional representation with

$Q_2 = u^{-1}b(t)\partial_x$ and $Q_1 = xu^{-1}b(t)\partial_x - b(t)\partial_u$. Then we must have $[Q_1, e_2] = 0$ and this gives us

$$[Q_1, e_2] = 3b(t)u^{-5}\partial_x = 0$$

which gives $b = 0$ and therefore we have no representation for $J = 1/2$.

For $J = 2$ we have $b = 0$ and we then have $Q_5 = c(t)u^5\partial_u$. Then we find, after a tedious calculation, that $(\mathrm{ad}e_2)^5 Q_5 = 0$, as required by the representation (both e_3 and e_2 act nilpotently and have nilpotent order $2J+1 = 5$), only if $c(t) = 0$ and thus there is no realization in this case. So this realization of $\mathrm{sl}(2,\mathbb{R})$ has an irreducible extension only for $J = 0$.

In the above calculations, we have shown that there are no irreducible representations of our realizations of $\mathrm{sl}(2,\mathbb{R})$ on solvable Lie algebras of dimensions greater than one. This means that all the representations we need to consider are direct sums of one-dimensional representations. Since each such representation requires that the vector fields of the representation space commutes with the vector fields of the realization of $\mathrm{sl}(2,\mathbb{R})$, it follows that we have to consider direct sums $\mathrm{sl}(2,\mathbb{R}) \oplus A$ where A is a solvable Lie algebra. We give these in the following section.

5.3 Admissible direct-sum extensions $\mathrm{sl}(2,\mathbb{R}) \oplus A$ of $\mathrm{sl}(2,\mathbb{R})$

Here we look at direct sum extensions $\mathrm{sl}(2,\mathbb{R}) \oplus A$ where A is a solvable Lie algebra: these are the reducible Lie algebras. The elements of A must then commute with those of $\mathrm{sl}(2,\mathbb{R})$, so we have that $A \subseteq \mathfrak{Z}$ where $\mathfrak{Z}$ is the commutator algebra of the appropriate realization of $\mathrm{sl}(2,\mathbb{R})$.

$\dim A = 1$:

We denote by Q the generic element of $\mathfrak{Z}$, and we write $\mathfrak{Z} = \langle Q \rangle$. We seek Q such that $[Q_i, Q] = 0$ for $i = 1, 2, 3$ where Q_1, Q_2, Q_3 are the generators of $\mathrm{sl}(2,\mathbb{R})$. Further, by equivalence transformations we mean those transformations of the equivalence group which leave invariant the forms of Q_1, Q_2, Q_3.

1. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t, -t^2\partial_t, \partial_t \rangle$.

   In this representation, we find that $F = 0$ which is inadmissible as a symmetry algebra, and so there are no admissible extensions.

2. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t \rangle$.

   We find that $\mathfrak{Z} = \langle c(u)\partial_u \rangle$ with $c(u) \neq 0$. The equivalence transformations are given by $t' = T(t) = t, x' = X = \pm x, u' = U = U(u)$ with $U(u)$ arbitrary so that $U'(u) \neq 0$.

   $Q = c(u)\partial_u$ is transformed to

   $$Q' = (QT)\partial_T + (QX)\partial_X + (QU)\partial_U = c(u)U'(u)\partial_U$$

and we then see that we may choose $U(u)$ with $c(u)U'(u) = 1$ so that $Q' = c(u)U'(u)\partial_U = \partial_U$. Hence we have the canonical form

$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle \oplus \langle \partial_u\rangle$$

for $\mathrm{sl}(2,\mathbb{R}) \oplus A$.

The coefficients of the corresponding invariant equation are

$$F = x^{-3}u_1^{-4}f(\sigma), \quad \sigma = \frac{u_2}{u_1^2} + \frac{3}{xu_1},$$

$$G = -\frac{x^{-2}\omega}{4} + x^{-2}\left(\frac{9\sigma}{\omega^2} - \frac{12}{\omega^3}\right) f(\sigma) + \frac{1}{x^2\omega}g(\sigma), \quad \omega = xu_1.$$

3. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle$.

   We find that $[Q_i, Q] = 0, \;\; i = 1, 2, 3$ gives $Q = b(x\partial_x + 2bu\partial_u) + c\partial_u$ where b, c are constants, so $\mathfrak{Z} = \langle \partial_u, x\partial_x + 2u\partial_u\rangle$.

   The equivalence transformations are of the form $t' = t, x' = ax, u' = a^2u+k$ with $a \neq 0, k$ constants. If we have $Q = bx\partial_x + (2bu+c)\partial_u$, then Q is transformed to $Q' = bX\partial_X + (2bU + c - 2bk)\partial_U$. If $b \neq 0$ then we may always choose $U = a^2u+k$ so that $c = 2bk$. This then gives us two canonical forms for Q: we have $Q = c\partial_u$ if $b = 0$ and $Q = b(x\partial_x + 2u\partial_u)$ if $b \neq 0$. Thus there are two canonical forms for Q and we have two possible one-dimensional extensions:

$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle \oplus \langle \partial_u\rangle$$
$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle \oplus \langle x\partial_x + 2u\partial_u\rangle.$$

   The coefficients F and G are given by

$$F = xf(\rho), \quad G = \frac{u_1^2}{4} + x^{-2}\tilde{g}(\rho), \quad \rho = x^2u_2 - xu_1,$$

$$F = f_0x\sigma, \quad G = \frac{u_1^2}{4} + x^{-2}\omega^2\tilde{g}(\omega/\sigma), \quad \omega = 2u - xu_1, \;\; \sigma = 2u - x^2u_2,$$

   where f_0 is a constant.

4. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle$.

   We find that $\mathfrak{Z} = \langle a(t)\partial_t + c(t,u)\partial_u\rangle$. The equivalence transformations are of the form $t' = T(t), x' = x, u' = U(t,u)$ with $\dot{T} \neq 0, U_u \neq 0$ and $Q = a(t)\partial_t + c(t,u)\partial_u$ is transformed into $Q' = a\dot{T}\partial_T + (aU_t + cU_u)\partial_U$. If $a \neq 0$ then we may choose T, U so that $a\dot{T} = 1$ and $a(t)U_t + c(t,u)U_u = 0$. This then gives the canonical form $Q = \partial_t$. If $a(t) = 0$ we choose U so that $c(t,u)U_u = 1$ and this gives us the canonical form $Q = \partial_t$. We then have the two canonical extensions

$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t\rangle,$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_u\rangle.$$

The coefficients F and G are

$$F = u_1^{-3} f(u), \quad G = -\frac{3}{2}\frac{u_2^2}{u_1^4} f(u) + g(u),$$

$$F = u_1^{-3} f(t), \quad G = -\frac{3}{2}\frac{u_2^2}{u_1^4} f(t) + g(t).$$

5. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_x + \partial_x, -t^2\partial_t - xt\partial_x, \partial_t\rangle$.

   This algebra is not an admissible symmetry algebra and so we have no admissible extensions.

6. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle$.

   We find $\mathfrak{Z} = \langle a(t)\partial_t + c(t)u\partial_u\rangle$. The equivalence transformations are given by $t' = T(t),\ x' = x,\ u' = \gamma(t)u$ with $\dot{T} \neq 0, \gamma \neq 0$. Then $Q = a(t)\partial_t + c(t)u$ is transformed to $Q' = a(t)\dot{T}\partial_T + (a(t)\dot{\gamma}u + c(t)\gamma(t)u)\partial_U$. If $a(t) \neq 0$ we choose T with $\dot{T} = 1/a(t)$ and $a(t)\dot{\gamma} + c(t)\gamma = 0$, and this gives the canonical form $Q = \partial_t$. If $a(t) = 0$ then we have $Q' = c(t)\gamma(t)u\partial_U = c(t)U\partial_U$. In this case, if $\dot{c} = 0$ we have the canonical form $Q = u\partial_u$; but if $\dot{c} \neq 0$ we choose $c(t) = T(t)$ and this gives us the canonical form $Q = tu\partial_u$.

   Thus we have the three canonical one-dimensional extensions:

$$\begin{aligned}&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle\partial_t\rangle,\\ &\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle u\partial_u\rangle\\ &\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle tu\partial_u\rangle.\end{aligned}$$

   The corresponding invariant equations have the coefficients F, G

$$F = u^{-6} f(\sigma), \quad G = ug(\sigma) + (12u^{-8}u_1^3 - 9u^{-7}u_1u_2) f(\sigma),$$

$$F = f(t)u^{-6}\sigma^{-3/2}, \quad G = ug(t) + f(t)(12u^{-8}u_1^3 - 9u^{-7}u_1u_2)\sigma^{-3/2},$$

$$F = f(t)u^{-6}\sigma^{-3/2}, \quad G = u\Big(-\frac{1}{4t}\log\sigma + g(t)\Big) + f(t)(12u^{-8}u_1^3 - 9u^{-7}u_1u_2)\sigma^{-3/2},$$

   where $\sigma = u^{-5}u_2 - 2u^{-6}u_1^2$.

7. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle$.

   We find $\mathfrak{Z} = \langle a(t)\partial_t\rangle$ and the equivalence transformations are $t' = T(t),\ x' = x,\ u' = \pm u$ with $\dot{T} \neq 0$. We find that $Q = a(t)\partial_t$ is transformed to $Q' = a(t)\dot{T}\partial_T$ and we choose T so that $\dot{T} = 1/a(t)$ and we obtain the canonical form $Q = \partial_t$. The canonical extension is then

$$\langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle\partial_t\rangle.$$

8. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -(u^{-4}+x^2)\partial_x + xu\partial_u, \partial_x\rangle$.

   We find $\mathfrak{Z} = \langle a(t)\partial_t\rangle$ and the equivalence transformations are $t' = T(t),\ x' = x,\ u' = \pm u$ with $\dot{T} \neq 0$. We find that $Q = a(t)\partial_t$ is transformed to $Q' = a(t)\dot{T}\partial_T$ and we choose T so that $\dot{T} = 1/a(t)$ and we obtain the canonical form $Q = \partial_t$. The canonical extension is then

$$\langle 2x\partial_x - u\partial_u, -(u^{-4}+x^2)\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle\partial_t\rangle.$$

   The invariant equations in cases 7. and 8. are obtained simply by restricting the arbitrary functions $f(t,\sigma)$ and $g(t,\sigma)$ of the last two cases in Section 3.3 to time independent ones.

In the following we sum up one-dimensional extensions of $\mathrm{sl}(2,\mathbb{R})$ in a list:

$$\begin{aligned}
&\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle \oplus \langle\partial_u\rangle\\
&\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle \oplus \langle\partial_u\rangle\\
&\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle \oplus \langle x\partial_x + 2u\partial_u\rangle\\
&\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle\partial_t\rangle,\\
&\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle\partial_u\rangle\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle\partial_t\rangle,\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle u\partial_u\rangle\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle tu\partial_u\rangle\\
&\langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle\partial_t\rangle\\
&\langle 2x\partial_x - u\partial_u, -(u^{-4} + x^2)\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle\partial_t\rangle.
\end{aligned}$$

$\dim A = 2$:

There are two canonical solvable Lie algebras of dimension 2: $A_{2.1} = \langle e_1, e_2\rangle,\ \ [e_1,e_2] = 0$ and $A_{2.2} = \langle e_1, e_2\rangle,\ \ [e_1,e_2] = e_1$. We begin with the one-dimensional extensions listed above

1. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle$.

   As noted above, the commutator algebra is $\mathfrak{Z} = \langle c(u)\partial_u\rangle\rangle$ with $c(u) \neq 0$ and there is only one canonical one-dimensional extension:

$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle \oplus \langle\partial_u\rangle.$$

   The equivalence group of this algebra is $T(t) = t,\ X = \pm x,\ U(u) = u + k,\ k \in \mathbb{R}$.

   We choose $e_1 = \partial_u$ and then we take $e_2 = c(u)\partial_u$. For $A_{2.1}$ we need $[e_1,e_2] = 0$ and this gives $c(u) =$ constant, which contradicts the requirement that $\dim A_{2.1} = 2$. Thus there is no such extension of $\mathrm{sl}(2,\mathbb{R})$. For $A_{2.2}$ we need $[e_1,e_2] = e_1$. This gives $c'(u) = 1$ which gives $c(u) = u+l,\ l \in \mathbb{R}$. Then we have $e_2 = (u+l)\partial_u$.

Under an equivalence transformation e_2 is transformed to $e_2' = (e_2U(u))\partial_U = (u+l)\partial_U = (U+l-k)\partial_U$; then choose $k = l$ and we find that $e_2' = U\partial_U$ and we obtain the canonical form $A_{2.2} = \langle \partial_u,\ u\partial_u\rangle$. Thus we obtain only one extension

$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle \oplus \langle \partial_u, u\partial_u\rangle.$$

2. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle$.

   The commutator algebra is $\mathfrak{Z} = \langle x\partial_x + 2u\partial_u, \partial_u\rangle$. It is evidently impossible to implement $A_{2.1}$ with $[e_1, e_2] = 0$ since the two canonical forms for e_1 are, as already shown, $e_1 = \partial_u$ and $e_1 = x\partial_x + 2u\partial_u$. Then, with $e_2 = a(x\partial_x + 2u\partial_u) + b\partial_u$, we find that $[\partial_u, e_2] = 0$ gives $a = 0$; further, $[x\partial_x + 2u\partial_u, e_2] = 0$ gives $b = 0$ and we then see that both these cases contradict the requirement that $\dim A_{2.1} = 2$. Hence there is no such extension by $A_{2.1}$. The algebra $A_{2.2}$ is implemented as $A_{2.2} = \langle \partial_u, x\partial_x + 2u\partial_u\rangle$. To see this, take $e_1 = \partial_u$, which is one of the canonical forms (see the one-dimensional extensions above). Then with $e_2 = a(x\partial_x + 2u\partial_u) + b\partial_u$ we have immediately that $\langle e_1, e_2\rangle = \langle \partial_u, x\partial_x + 2u\partial_u\rangle$. The same result is obtained with the other canonical form $e_1 = x\partial_x + 2u\partial_u$. Thus we obtain only one two-dimensional extension:

$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle \oplus \langle \partial_u, x\partial_x + 2u\partial_u\rangle.$$

3. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle$.

   The commutator algebra is $\mathfrak{Z} = \langle Q\rangle = \langle a(t)\partial_t + c(t,u)\partial_u\rangle$, and we then have the two canonical one-dimensional extensions

$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t\rangle,$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_u\rangle.$$

   We begin with the algebra

$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t\rangle.$$

   The equivalence group of this extension is $t' = t + k, x' = x, u' = U(u)$. We take $e_1 = \partial_t$ and we seek $e_2 = a(t)\partial_t + c(t,u)\partial_u$ so as to find a two-dimensional extension. For $A_{2.1}$ we seek e_2 with $[e_1, e_2] = 0$ and this gives $a =$ constant, $c = c(u)$, which gives us $A_{2.1} = \langle \partial_t, c(u)\partial_u\rangle$. Thus we take $e_2 = c(u)\partial_u$ with $c(u) \neq 0$. Under an equivalence transformation, e_2 is transformed to $e_2' = c(u)U'(u)\partial_U$ and we choose $U(u)$ so that $c(u)U'(u) = 1$. This gives the canonical form $A_{2.1} = \langle \partial_t, \partial_u\rangle$.

   For $A_{2.2}$ we seek e_2 with $[e_1, e_2] = e_1$ and this gives $\dot{a} = 1$, $c_t = 0$ so that $a(t) = t + l$, $(l \in \mathbb{R})$, $c = c(u)$, so $e_2 = (t+l)\partial_t + c(u)\partial_u$. Under an equivalence transformation, e_2 is mapped to $e_2' = (t+l)\partial_T + c(u)U'(u)\partial_U$. Clearly, we choose $T = t + k$ so that $t + l = T + l - k = T$ by taking $k = l$ and then we find $e_2' = T\partial_T + c(u)U'(u)\partial_U$. If $c(u) = 0$ then we find $e_2' = T\partial_T$, whereas if $c(u) \neq 0$,

we choose $U(u)$ so that $c(u)U' = U$ and we find $e_2' = T\partial_T + U\partial_U$. Thus, we find the two canonical forms for $A_{2.2} = \langle \partial_t, t\partial_t \rangle$ and $A_{2.2} = \langle \partial_t, t\partial_t + u\partial_u \rangle$.

If we now take the extension

$$\langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle \oplus \langle \partial_u \rangle$$

we find that its equivalence group is $t' = T(t)$, $\dot{T} \neq 0$, $X = x$, $U(t,u) = u + k(t)$. We take $e_1 = \partial_u$ and we seek $e_2 = a(t)\partial_t + c(t,u)\partial_u$. For $A_{2.1}$ we want $[e_1, e_2] = 0$ which gives $c_u = 0$ so $e_2 = a(t)\partial_t + c(t)\partial_u$. Under an equivalence transformation, e_2 is mapped to $e_2' = a(t)\dot{T}\partial_T + (a(t)\dot{k}(t) + c(t))\partial_U$. There are two cases to consider: $a = 0$ and $a \neq 0$. If $a \neq 0$ then we may always choose $k(t)$ so that $a(t)\dot{k}(t) + c(t) = 0$ and we choose T so that $a(t)\dot{T}(t) = 1$, thus giving $e_2' = \partial_T$. If $a = 0$ then we must have $c(t) \neq$ constant so as to avoid $\dim_{\mathbb{R}}\langle e_1, e_2 \rangle = 1$. Thus, we want $\dot{c}(t) \neq 0$ and we may then choose $T = c(t)$ to obtain $e_2' = T\partial_U$. Thus we have two canonical forms: $A_{2.1} = \langle \partial_u, \partial_t \rangle$ and $A_{2.1} = \langle \partial_u, t\partial_u \rangle$. However, the second algebra is not a symmetry algebra of our equation (it leads to the contradiction $u = 0$), so we find only one admissible canonical form $A_{2.1} = \langle \partial_u, \partial_t \rangle$.

For $A_{2.2}$ we want $[e_1, e_2] = e_1$ and this gives $c_u = 1$ which gives $c(t,u) = u + l(t)$. Thus $e_2 = a(t)\partial_t + (u + l(t))\partial_u$. Under and equivalence transformation, e_2 is transformed to $e_2' = a(t)\dot{T}\partial_T + (a(t)\dot{k}(t) + u + l(t))\partial_U$. If $a = 0$ then we choose $U = u + k(t)$ so that $u + l(t) = U + l(t) - k(t) = U$ by taking $k(t) = l(t)$. This gives $e_2' = U\partial_U$. If $a \neq 0$ then we choose T so that $a(t)\dot{T} = T$ and we choose $k(t)$ so that $a(t)\dot{k}(t) + u + l(t) = U + a(t)\dot{k}(t) + l(t) - k(t) = U$ by taking $k(t) = l(t)$. This gives $e_2' = T\partial_T + U\partial_U$, and we find the two canonical forms $A_{2.2} = \langle \partial_u, u\partial_u \rangle$ and $A_{2.2} = \langle \partial_u, t\partial_t + u\partial_u \rangle$. Thus we have the following canonical two-dimensional extensions:

$$\begin{aligned}
&\langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle \oplus \langle \partial_t, t\partial_t \rangle,\\
&\langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle \oplus \langle \partial_t, t\partial_t + u\partial_u \rangle,\\
&\langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle \oplus \langle \partial_t, \partial_u \rangle,\\
&\langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle \oplus \langle \partial_u, u\partial_u \rangle,\\
&\langle 2x\partial_x, -x^2\partial_x, \partial_x \rangle \oplus \langle \partial_u, t\partial_t + u\partial_u \rangle.
\end{aligned}$$

4. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x \rangle$.

   The commutator algebra is $\mathfrak{Z} = \langle a(t)\partial_t + c(t)u\partial_u \rangle$. The equivalence transformations are given by $t' = T(t)$, $x' = x$, $u' = \gamma(t)u$ with $\dot{T} \neq 0, \gamma \neq 0$. We have the three canonical extensions

$$\begin{aligned}
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x \rangle \oplus \langle \partial_t \rangle,\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x \rangle \oplus \langle u\partial_u \rangle\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x \rangle \oplus \langle tu\partial_u \rangle.
\end{aligned}$$

For the extension

$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t\rangle$$

we take $e_1 = \partial_t$. The equivalence group of this extension is $t' = t + k$, $k \in \mathbb{R}$, $x' = x$, $u' = \gamma u$, $\gamma \in \mathbb{R}, \gamma \neq 0$. For $A_{2.1}$ we seek $e_2 = a(t)\partial_t + c(t)u$ with $[\partial_t, e_2] = 0$ which gives $a, c =$ constants. So $e_2 = a\partial_t + cu\partial_u$, and we find that $A_{2.1} = \langle \partial_t, u\partial_u\rangle$. For $A_{2.2}$ we need $e_2 = a(t)\partial_t + c(t)u$ with $[\partial_t, e_2] = \partial_t$ which gives $\dot{a} = 1, c =$ constant and we then have $a = t+l, l \in \mathbb{R}$. Under an equivalence transformation, e_2 is mapped to $e_2' = (t+l)\partial_T + c\gamma u\partial_U = (T+l-k)\partial_T + cU\partial_U$. Choose $k = l$ and we have $e_2' = T\partial_T + cU\partial_U$ for arbitrary $c \in \mathbb{R}$, giving the canonical form $A_{2.2} = \langle \partial_t, t\partial_t + cu\partial_u\rangle$ where $c \in \mathbb{R}$ is arbitrary.

For the extension

$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle u\partial_u\rangle$$

we take $e_1 = u\partial_u$. The equivalence group of this extension is $t' = T(t)$, $\dot{T} \neq 0$, $x' = x, u' = \gamma(t)u$, $\gamma(t) \neq 0$. For $A_{2.1}$ we want $e_2 = a(t)\partial_t + c(t)u\partial_u$ with $[u\partial_u, e_2] = 0$, which is automatically satisfied. Under an equivalence transformation, e_2 is mapped to $e_2' = a(t)\dot{T}\partial_T + (a(t)\dot{\gamma}(t) + c(t))u\partial_U$. If $a = 0$ then we cannot have $c =$ constant in order to avoid $\dim_{\mathbb{R}}\langle e_1, e_2\rangle = 1$. Thus we have $\dot{c}(t) \neq 0$, and we take $T(t) = c(t)$ giving the canonical form $e_2 = tu\partial_u$. If $a \neq 0$, we choose T so that $a(t)\dot{T} = 1$ and we choose $\gamma(t)$ so that $a(t)\dot{\gamma}(t) + c(t) = 0$, which gives $e_2' = \partial_T$. Thus we have the canonical forms $A_{2.1} = \langle u\partial_u, \partial_t\rangle$ and $A_{2.1} = \langle u\partial_u, tu\partial_u\rangle$. However, if $c(t)u\partial_u$ is to be a symmetry, then we find, after routine calculations, that

$$uG_u + u_1G_{u_1} + u_2G_{u_2} = G + \frac{\dot{c}(t)}{c(t)}u,$$

which gives a contradiction if we have both $u\partial_u$ and $tu\partial_u$ as symmetries. Hence only $A_{2.1} = \langle u\partial_u, \partial_t\rangle$ is admissible.

For $A_{2.2}$ we want $[e_1, e_2] = e_1$ which is not possible since we have $[e_1, e_2] = 0$.

We now look at the extension

$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle tu\partial_u\rangle.$$

The equivalence algebra is $t' = t$, $x' = x$, $u' = \gamma(t)u$, $\gamma \neq 0$. For $A_{2.1}$ we take $e_1 = tu\partial_u$ and we seek $e_2 = a(t)\partial_t + c(t)u\partial_u$ with $[e_1, e_2] = 0$, which gives $a(t) = 0$. Clearly we need $c(t) \neq 0$ and we find, after routine calculations, that e_2 leads to the condition

$$uG_u + u_1G_{u_1} + u_2G_{u_2} = G + \frac{\dot{c}(t)}{c(t)}u,$$

whereas $e_1 = tu\partial_u$ gives (on replacing $c(t)$ with t)

$$uG_u + u_1G_{u_1} + u_2G_{u_2} = G + \frac{u}{t},$$

so we see that only $t\dot{c}(t) = c(t)$ is possible, which means that $c(t) = kt,\ k \in \mathbb{R}$. However, this gives $\dim_{\mathbb{R}}\langle e_1, e_2\rangle = 1$, so we obtain no extension by $A_{2.1}$.

For $A_{2.2}$ we seek e_2 with $[e_1, e_2] = e_1$. This gives $a(t) = -t$ and we have $e_2 = -t\partial_t + c(t)\partial_u$. Under an equivalence transformation $t' = t$, $x' = x$, $u' = \gamma(t)u$, $\gamma \neq 0$ we find that e_2 is mapped to $e_2' = -T\partial_T + (-t\dot{\gamma}(t) + c(t)\gamma(t))u\partial_U$, from which we see that on choosing $\gamma(t)$ so that $t\dot{\gamma}(t) + \gamma(t)c(t) = 0$, we obtain $e_2' = -T\partial_T$. From this it follows that we have the canonical extension

$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle tu\partial_u, -t\partial_t\rangle.$$

Thus we have three two-dimensional extensions:

$$\begin{aligned}
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, u\partial_u\rangle,\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, t\partial_t + cu\partial_u\rangle, \quad c \in \mathbb{R},\\
&\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle tu\partial_u, -t\partial_t\rangle.
\end{aligned}$$

5. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle$.

   The commutator algebra is $\mathfrak{Z} = \langle a(t)\partial_t\rangle$. There is one canonical one-dimensional extension:
   $$\langle 2x\partial_x - u\partial_u, (u^{-4} - x^2)\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t\rangle.$$
   its equivalence group is $t' = t + k$, $k \in \mathbb{R}$, $x' = x, u' = \pm u$. We take $e_1 = \partial_t$. For $A_{2.1}$ we want $e_2 = a(t)\partial_t$ with $[\partial_t, e_2] = 0$ and this implies $a =$ constant which we discard since this gives us $\dim_{\mathbb{R}}\langle e_1, e_2\rangle = 1$. Therefore there is no extension by $A_{2.1}$. For $A_{2.2}$ we want $[\partial_t, e_2] = \partial_t$, which gives $\dot{a} = 1$, from which we have $a(t) = t + l$, $l \in \mathbb{R}$. Under an equivalence transformation, e_2 is mapped to $e_2' = (t + l)\partial_T = (T + l - k)\partial_T = T\partial_T$ on choosing $k = l$. Thus we have the canonical form $e_2 = t\partial_t$ and $A_{2.2} = \langle \partial_t, t\partial_t\rangle$. However, this is not an admissible extension since requiring $\langle \partial_t, t\partial_t\rangle$ to be a symmetry implies $F = 0$, which is not allowed. So we have no extension.

6. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -(u^{-4} + x^2)\partial_x + xu\partial_u, \partial_x\rangle$.

   The commutator algebra is $\mathfrak{Z} = \langle a(t)\partial_t\rangle$, as above. There is only one canonical one-dimensional extension:
   $$\langle 2x\partial_x - u\partial_u, -(u^{-4} + x^2)\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t\rangle.$$
   Its equivalence group is $t' = t + k$, $k \in \mathbb{R}$, $x' = x, u' = \pm u$. As in the previous case, the only possible extension we obtain is $A_{2.2} = \langle \partial_t, t\partial_t\rangle$ which is not an admissible extension since requiring $\langle \partial_t, t\partial_t\rangle$ as it gives $F = 0$, which is not allowed. So we have no extension.

We then have the following list of admissible two-dimensional extensions $\mathrm{sl}(2,\mathbb{R}) \oplus A$ where A is a solvable Lie algebra:

$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle \oplus \langle \partial_u, u\partial_u\rangle$$
$$\langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle \oplus \langle \partial_u, x\partial_x + 2u\partial_u\rangle$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t, t\partial_t\rangle,$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t, t\partial_t + u\partial_u\rangle,$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t, \partial_u\rangle,$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_u, u\partial_u\rangle,$$
$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_u, t\partial_t + u\partial_u\rangle$$
$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, u\partial_u\rangle,$$
$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, t\partial_t + cu\partial_u\rangle, \quad c \in \mathbb{R},$$
$$\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle tu\partial_u, -t\partial_t\rangle.$$

Invariant Equations: Among above realizations only seven admits invariant equations that involve two constants f_0, g_0 rather than arbitrary functions. They are given by the following:

1. $\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t, \partial_u\rangle$;
   $$F = f_0 u_1^{-3}, \quad G = -f_0 \frac{3}{2}\frac{u_2^2}{u_1^4} + g_0$$
2. $\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t, t\partial_t + u\partial_u\rangle$
   $$F = f_0 u^2 u_1^{-3}, \quad G = -\frac{3}{2} f_0 \frac{u^2 u_2^2}{u_1^4} + g_0,$$
3. $\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_u, t\partial_t + u\partial_u\rangle$
   $$F = f_0 t^2 u_1^{-3}, \quad G = -\frac{3}{2} f_0 t^2 \frac{u_2^2}{u_1^4} + g_0,$$
4. $\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, u\partial_u\rangle$;
   $$F = f_0 u^{-6}\sigma^{-3/2}, \quad G = g_0 u + f_0(12u^{-8}u_1^3 - 9u^{-7}u_1u_2)\sigma^{-3/2},$$
5. $\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, t\partial_t + cu\partial_u\rangle$, $c \in \mathbb{R}$;
   $$F = f_0 u^{-6}\sigma^{(1-6c)/(4c)}, \quad G = g_0 u\sigma^{1/(4c)} + f_0(12u^{-8}u_1^3 - 9u^{-7}u_1u_2)\sigma^{(1-6c)/4c},$$
6. $\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle t\partial_t, tu\partial_u\rangle$;
   $$F = f_0 t^{-1}u^{-6}\sigma^{-3/2}, \quad G = t^{-1}[u\Big(-\frac{1}{4}\log\sigma + g_0\Big) + f_0(12u^{-8}u_1^3 - 9u^{-7}u_1u_2)\sigma^{-3/2}],$$
7. $\langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle \oplus \langle t\partial_t, tu\partial_u\rangle$;
   $$F = f_0 t^{-1}u^{-6}\sigma^{-3/2}, \quad G = g_0 t^{-1}u + f_0 t^{-1}(12u^{-8}u_1^3 - 9u^{-7}u_1u_2)\sigma^{-3/2},$$

where $\sigma = u^{-6}u_2 - 2u^{-6}u_1^2$.

Remark: Note that the first equation is within class (1.7). By an exchange of variables $(x, u) \to (u, x)$, it is equivalent to the case $F = 1$. We mention that the Lie point symmetry algebra of the standard Harry-Dym equation (1.5) has the direct-sum structure
$$L = \langle x\partial_x + u\partial_u, x^2\partial_x + 2xu\partial_u, \partial_x\rangle \oplus \langle \partial_t, t\partial_t - \frac{u}{3}\partial_u\rangle.$$

It is easy to see that L is isomorphic to the fifth realization in the above list. This is readily achieved by choosing $c = 1/3$ and applying the transformation $u = \tilde{u}^{-1/2}$.

$\dim A = 3$:

We now turn to three-dimensional solvable Lie algebras. There are eleven such inequivalent types (given in the appendix). We treat each realization of $\mathrm{sl}(2,\mathbb{R})$ separately.

1. $\langle 2t\partial_t + x\partial_x, -t^2\partial_t + (x^3 - xt)\partial_x, \partial_t\rangle$.

   The commutator is $\mathfrak{Z} = \langle c(u)\partial_u\rangle\rangle$ with $c(u) \neq 0$. In the canonical forms for $A_{3.n},\ n = 1, \ldots, 11$ we have either $[e_1, e_2] = 0$ or $[e_1, e_3] = 0$. On choosing e_1 in the canonical form $e_1 = \partial_u$ and $e_2 = c(u)\partial_u$ we have $c'(u) = 0$ so that $c =$ constant and $e_2 = ce_1$. The same is true for e_3 if $[e_1, e_3] = 0$. From this it follows that $\dim_{\mathbb{R}} A_{3.n} \leq 2$ which is a contradiction. Thus there are no three-dimensional direct-sum extensions of this realization of $\mathrm{sl}(2,\mathbb{R})$.

2. $\mathrm{sl}(2,\mathbb{R}) = \langle 2t\partial_t + x\partial_x, -t^2\partial_t - xt\partial_x + x^2\partial_u, \partial_t\rangle$.

   The commutator is $\mathfrak{Z} = \langle \partial_u, x\partial_x + 2u\partial_u\rangle$, which is a two-dimensional Lie algebra, so it is impossible to realize $A_{3.n}$ using vector fields from this algebra, and there are consequently no three-dimensional direct-sum extensions of this realization of $\mathrm{sl}(2,\mathbb{R})$.

3. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle$.

   The commutator is $\mathfrak{Z} = \langle a(t)\partial_t + c(t,u)\partial_u\rangle$, and the equivalence group is $t' = T(t),\ \dot{T} \neq 0,\ x' = x,\ u' = U(t,u)$. There are two canonical forms for e_1: $e_1 = \partial_t$ and $e_1 = \partial_u$. $A_{3.1}$ requires $[e_i, e_j] = 0,\ i,j = 1,2,3$. Putting $e_1 = \partial_t$ we may then choose $e_2 = \partial_u$ and then $e_3 = a\partial_t + c\partial_u,\ a, c \in \mathbb{R}$ follows from the commutation relations, so that $\dim_{\mathbb{R}}\langle e_1, e_2, e_3\rangle = 2$, which is not possible. The same occurs for $e_1 = \partial_u$: we choose $e_2 = \partial_t$ and $e_3 = a\partial_t + c\partial_u,\ a, c \in \mathbb{R}$.

   $\underline{A_{3.2} = A_{2.2} \oplus A_1 : [e_1, e_2] = e_1,\ [e_1, e_3] = [e_2, e_3] = 0}$. To implement $[e_1, e_2] = e_1$ we may choose among the following two-dimensional extensions:

   $$\begin{array}{llll} (i) & e_1 = \partial_t, e_2 = t\partial_t & (ii) & e_1 = \partial_u, e_2 = u\partial_u, \\ (iii) & e_1 = \partial_t, e_2 = t\partial_t + u\partial_u, & (iv) & e_1 = \partial_u, e_2 = t\partial_t + u\partial_u. \end{array}$$

   However, the first two cases (i), (ii) give $F = 0$ so they are inadmissible. Thus we begin with $e_1 = \partial_t, e_2 = t\partial_t + u\partial_u$. The commutation relations $[e_1, e_3] = 0,\ [e_2, e_3] = 0$ give $e_3 = ku\partial_u,\ k \in \mathbb{R}$ so we find the extension $A_{3.2} = \langle \partial_t, t\partial_t, u\partial_u\rangle$. But if $\partial_t,\ t\partial_t$ are symmetries, then we find that $F = 0$, so this realization is not allowed.

   Next we take up $e_1 = \partial_u, e_2 = t\partial_t + u\partial_u$. The commutation relations give $e_3 = kt\partial_t + lt\partial_u,\ k, l \in \mathbb{R}$. If $k = 0$ we have $A_{3.2} = \langle \partial_u, t\partial_t + u\partial_u, t\partial_u\rangle$. If $k \neq 0$ then we use an equivalence transformation leaving $\mathrm{sl}(2,\mathbb{R})$ and e_1, e_2

invariant in form: $t' = \alpha t$, $(\alpha \neq 0)$, $x' = x$, $u' = u + \beta t$. Under such an equivalence transformation e_3 is mapped to $e_3' = kT\partial_T + t(l + \beta k)\partial_u$ and we then choose β so that $l + \beta k = 0$ so we find $e_3' = kT\partial_T$ and we obtain the canonical form $A_{3.2} = \langle \partial_u, u\partial_u, t\partial_t \rangle$. We note that if the vector field $c(t)\partial_u$ is a symmetry, then we obtain $c(t)G_u = \dot{c}(t)$. From this it follows that we cannot have both ∂_u and $t\partial_u$ as symmetries. Also, we know that $F = u_1^{-3}f(t,u)$. If ∂_u is a symmetry, then $F_u = 0$ and $F = u_1^{-3}f(t)$. Then if $u\partial_u$ is a symmetry, we have $uF_u + u_1F_{u_1} + u_2F_{u_2} = 0$ and so we obtain $u_1F_{u_1} = 0$ which is possible only if $f(t) = 0$, contradicting the requirement that $F \neq 0$. Thus we have no admissible direct-sum extension by $A_{3.2}$

$A_{3.3} : [e_2, e_3] = e_1,\ [e_1, e_2] = [e_1, e_3] = 0$. There are only two admissible realizations with $[e_1, e_2] = 0$: $e_1 = \partial_t,\ e_2 = \partial_u$ and $e_1 = \partial_u,\ e_2 = \partial_t$. If $e_1 = \partial_t,\ e_2 = \partial_u$ then, with $e_3 = a(t)\partial_t + c(t,u)\partial_u$, we need $[e_2, e_3] = e_1$, which gives $c_u\partial_u = \partial_t$ which is obviously impossible. Then, with $e_1 = \partial_u,\ e_2 = \partial_t$, we find that $[e_2, e_3] = e_1$ gives $\dot{a}\partial_t + c_t\partial_u = \partial_u$, which implies $a =$ constant and $c = t + b(u)$. The relation $[e_1, e_3] = 0$ then gives $b(u) =$ constant so we have $e_3 = a_t + (t + b)\partial_u,\ a, b \in \mathbb{R}$, and we find the realization of $A_{3.3} = \langle \partial_u, \partial_t, t\partial_u \rangle$, which is not admissible because, as we have seen above, ∂_u and $t\partial_u$ cannot be symmetries simultaneously as they lead to the contradiction $G_u = 0,\ tG_u = 1$.

$A_{3.4} : [e_1, e_3] = e_1,\ [e_1, e_2] = 0,\ [e_2, e_3] = e_1 + e_2$. We have only two realizations of $[e_1, e_3] = e_1$: $e_1 = \partial_t,\ e_3 = t\partial_t + u\partial_u$ and $e_1 = \partial_u,\ e_3 = t\partial_t + u\partial_u$. With $e_1 = \partial_t,\ e_3 = t\partial_t + u\partial_u$ and $[e_1, e_2] = 0,\ [e_2, e_3] = e_1 + e_2$ we find that $\partial_t + uc_u\partial_u = 0$ which is impossible. Then with $e_1 = \partial_u,\ e_3 = t\partial_t + u\partial_u$, we find that $[e_1, e_2] = 0$ gives $e_2 = a(t)\partial_t + c(t)\partial_u$. The relation $[e_2, e_3] = e_1 + e_2$ then leads to $(a - t\dot{a})\partial_t + (c - t\dot{c}\partial_u = a\partial_t + (1 + c)\partial_u$, from which we find $a =$ constant and $c(t) = -\ln|t| + b,\ b \in \mathbb{R}$. So $e_2 = a\partial_t + (-\ln|t| + b)\partial_u$. If $a = 0$ then we obtain $A_{3.4} = \langle \partial_u, -\ln|t|\partial_u, t\partial_t + u\partial_u \rangle$. However, the first two vector fields lead to the contradiction $G_u = 0,\ t\ln|t|G_u = 1$, so this realization is not admissible. If, however $a \neq 0$ then we apply an equivalence transformation $t' = \alpha t$, $(\alpha \neq 0)$, $x' = x$, $u' = u + \beta t$ (which leaves the realization of $\mathrm{sl}(2, \mathbb{R})$ and e_1, e_2 invariant in form), and then e_2 is mapped to $e_2' = a\alpha\partial_t + (b + \alpha\beta - \ln|t|)\partial_u$ and then we may choose α, β so that $a\alpha = 1,\ b + \alpha\beta = 0$, and in this way we obtain the canonical form $e_2 = \partial_t - \ln|t|\partial_u$. So we have $A_{3.4} = \langle \partial_u, \partial_t - \ln|t|\partial_u, t\partial_t + u\partial_u \rangle$. The first two vector fields lead to $F_u = F_t = 0$, so we have $F = Ku_1^{-3}$ where $K \in \mathbb{R}$). The third vector field gives $uF_u + u_1F_{u_1} + u_2F_{u_2} + F = 0$, which is only possible if $K = 0$, a contradiction. Thus we have no admissible realization of $A_{3.4}$.

$A_{3.5} : [e_1, e_3] = e_1,\ [e_1, e_2] = 0,\ [e_2, e_3] = e_2$. First we have $e_1 = \partial_t,\ e_3 = t\partial_t + u\partial_u$, and then $[e_1, e_2] = 0$ gives $e_2 = a\partial_t + c(u)\partial_u,\ a \in \mathbb{R}$, and then $[e_2, e_3] = e_2$ gives $uc'(u) = 0$ so $c \in \mathbb{R}$ and we then have $e_2 = a\partial_t + c\partial_u,\ a, c \in \mathbb{R}$ so that we obtain $A_{3.5} = \langle \partial_t, \partial_u, t\partial_t + u\partial_u \rangle$. Then we take $e_1 = \partial_u,\ e_3 = t\partial_t + u\partial_u$ and the commutation relations give $a, c \in \mathbb{R}$ so we find $A_{3.5} = \langle \partial_u, \partial_t, t\partial_t + u\partial_u \rangle$. Thus we find only one realization of $A_{3.5} = \langle \partial_t, \partial_u, t\partial_t + u\partial_u \rangle$. With $F = u_1^{-3}f(t,u)$ the first two lead to $f(t,u) = K \in \mathbb{R}$ and the third vector field leads

to $uF_u+u_1F_{u_1}+u_2F_{u_2}+F=0$, which is only possible if $K=0$, a contradiction.

$A_{3.6}:[e_1,e_3]=e_1,\ [e_1,e_2]=0,\ [e_2,e_3]=-e_2$. Choosing first $e_1=\partial_t,\ e_3=t\partial_t+u\partial_u$, the commutation relations give $a=0$ and $c(t,u)=c(u)$ with $uc'(u)=2c(u)$ from which we deduce that $c(u)=bu^2,\ b\in\mathbb{R}$. Thus $e_2=bu^2\partial_u$ and we have $A_{3.6}=\langle\partial_t,u^2\partial_u,t\partial_t+u\partial_u\rangle$.

Then we take $e_1=\partial_u,\ e_3=t\partial_t+u\partial_u$ and the commutation relations give $a=a(t),c=c(t)$ with $t\dot{a}=2a,\ t\dot{c}=2c$, so that $a=kt^2,\ c=lt^2,\ k,l\in\mathbb{R}$. Hence $e_2=kt^2\partial_t+lt^2\partial_u$. If $k=0$ then we find that $A_{3.6}=\langle\partial_u,t^2\partial_u,t\partial_t+u\partial_u\rangle$. If $k\neq0$ then we apply an equivalence transformation $t'=\alpha t,\ (\alpha\neq0),\ x'=x,u'=u+\beta t$ (preserving $\mathrm{sl}(2,\mathbb{R})$ and e_1,e_3) so that e_2 is mapped to $e_2'=kt^2\partial_T+t^2(l+k\beta)\partial_U$ and we then choose $\alpha=1$ and β so that $l+k\beta=0$ so that $e_2'=kt^2\partial_T$ so we can take the canonical form $e_2=t^2\partial_t$. Thus we have $A_{3.6}=\langle\partial_u,t^2\partial_t,t\partial_t+u\partial_u\rangle$. We then have the following realizations of $A_{3.6}$:

$$\begin{aligned}A_{3.6}&=\langle\partial_t,u^2\partial_u,t\partial_t+u\partial_u\rangle\\A_{3.6}&=\langle\partial_u,t^2\partial_u,t\partial_t+u\partial_u\rangle\\A_{3.6}&=\langle\partial_u,t^2\partial_t,t\partial_t+u\partial_u\rangle.\end{aligned}$$

We have $F=f(t,u)u_1^{-3}$. In the first realization, $\partial_t,\ u^2\partial_u$ as symmetries give $f=Ku^6u_1^{-3}$. The third symmetry gives $uF_u+u_1F_{u_1}+u_2F_{u_2}+F=0$, which is only possible if $K=0$, a contradiction. In the second, we obtain the contradiction $G_u=0,\ t_u^G=2t$ from the first two vector fields. In the third realization, the first two vector fields give $F_u=0,\ tF_t+2F=0$. The third vector field gives $uF_u+tF_t+u_1F_{u_1}+u_2F_{u_2}+F=0$. These conditions on $F=f(t,u)u_1^{-3}$ require $f(t,u)=0$, a contradiction. So we have no admissible realization.

$A_{3.7}:[e_1,e_3]=e_1,\ [e_1,e_2]=0,\ [e_2,e_3]=qe_2,\ 0<|q|<1$. We have two canonical choices for implementing $e_1,\ e_2$: either $e_1=\partial_t,\ e_2=\partial_u$ or $e_1=\partial_u,\ e_2=\partial_t$. In the first case, we take $e_1=\partial_t,\ e_2=\partial_u,\ e_3=a(t)\partial_t+c(t,u)\partial_u$. Then $e[e_1,e_3]=e_1$ gives $\dot{a}=1,\ c_t=0$ so we have $e_3=(t+k)\partial_t+c(u)\partial_u$. Then $[e_2,e_3]=qe_2$ gives $c'(u)=q$ so that we find $e_3=(a+k)\partial_t+(qu+l)\partial_u$ with $k,l\in\mathbb{R}$. Consequently, we find that $A_{3.7}=\langle\partial_t,\partial_u,t\partial_t+qu\partial_u\rangle,\ 0<|q|<1$. If, on the other hand, $e_1=\partial_u,\ e_2=\partial_t$, similar calculations give $A_{3.7}=\langle\partial_t,\partial_u,qt\partial_t+u\partial_u\rangle,\ 0<|q|<1$. We have $F=u_1^{-3}f(t,u)$ so that in both cases the first two vector fields give $f(t,u)=K\in\mathbb{R}$. If the vector field $t\partial_t+qu\partial_u$ is a symmetry of our equation, then we find that $(1-3q)Ku_1^{-3}=0$ so that we are allowed $F\neq0$ only for $q=1/3$, which satisfies $0<|q|<1$. However, if $qt\partial_t+u\partial_u$ is to be a symmetry, then $(3-q)Ku_1^{-3}=0$, which gives $K=0$ as $0<|q|<1$, and this is a contradiction. Thus we find only one admissible algebra:

$$A_{3.7}=\langle\partial_t,\partial_u,t\partial_t+\frac{u}{3}\partial_u\rangle.$$

$A_{3.8}:[e_1,e_3]=-e_2,\ [e_1,e_2]=0,\ [e_2,e_3]=e_1$. There are only two realizations of $[e_1,e_2]=0$: $e_1=\partial_t,e_2=\partial_u$ and $e_1=\partial_u,e_2=\partial_t$. For $e_1=\partial_t,e_2=\partial_u$, and

with $e_3 = a(t)\partial_t + c(t,u)\partial_u$, the relation $[e_1, e_3] = e_2$ gives $\dot{a} = 0,\ c_t = 0$ so that $a =$ constant and $c = c(u)$. This gives $e_3 = a\partial_t + c(u)\partial_u$, and then $[e_2, e_3] = e_1$ gives $c'(u) = 0$ so that $c =$ constant and it follows that $\dim_{\mathbb{R}}\langle e_1, e_2, e_3\rangle = 2$, which is not possible, so we have no realization of $A_{3.8}$ in this case. For $e_1 = \partial_u,\ e_2 = \partial_t$ the same result is obtained. Thus, $A_{3.8}$ is not implemented as a direct-sum extension of this realization of $\mathrm{sl}(2,\mathbb{R})$.

$\underline{A_{3.9} : [e_1, e_3] = qe_1 - e_2,\ [e_1, e_2] = 0,\ [e_2, e_3] = e_1 + qe_2,\ q > 0}$. Again, either $e_1 = \partial_t, e_2 = \partial_u$ or $e_1 = \partial_u, e_2 = \partial_t$. If $e_3 = a(t)\partial_t + c(t,u)\partial_u$ then $[e_2, e_3] = e_1 + qe_2$ gives $c_u\partial_u = \partial_t + q\partial_u$ which is impossible. If $e_1 = \partial_u, e_2 = \partial_t$, then $[e_1, e_3] = qe_1 - e_2$ gives $c_u\partial_u = q\partial_u - \partial_t$ which is also impossible. Thus, there is no realization of $A_{3.9}$.

4. $\mathrm{sl}(2,\mathbb{R}) = \langle 2x\partial_x - u\partial_u, -x^2\partial_x + xu\partial_u, \partial_x\rangle$.

$\underline{A_{3.1} : [e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0}$. From the previous analysis, we know that we may choose either $e_1 = \partial_t,\ e_2 = u\partial_u$ or $e_1 = u\partial_u,\ e_2 = t\partial_t$. If we have $e_1 = \partial_t,\ e_2 = u\partial_u,\ e_3 = a(t)\partial_t + c(t)u\partial_u$ then the relation $[e_1, e_3] = 0$ gives $\dot{a} = \dot{c} = 0$, whence $e_3 = ae_1 + ce_2,\ a, c \in \mathbb{R}$, so that $\dim_{\mathbb{R}}\langle e_1, e_2, e_3\rangle = 2$, which is not possible. If we now have $e_1 = u\partial_u,\ e_2 = t\partial_t,\ e_3 = a(t)\partial_t + c(t)u\partial_u$, then $[e_1, e_3] = 0$ automatically and $[e_2, e_3] = 0$ gives $t\dot{a} = a,\ t\dot{c} = 0$, which implies that $a = kt, k \in \mathbb{R},\ c \in \mathbb{R}$ and so $e_3 = ke_1 + ce_2$, giving $\dim_{\mathbb{R}}\langle e_1, e_2, e_3\rangle = 2$, which is not possible.

$\underline{A_{3.2} : [e_1, e_2] = e_1,\ [e_1, e_3] = [e_2, e_3] = 0}$. In this case, to implement $[e_1, e_2] = e_1$ we may choose either $e_1 = \partial_t,\ e_2 = t\partial_t + ku\partial_u,\ k \in \mathbb{R}$ or $e_1 = tu\partial_u,\ e_2 = -t\partial_t$. If we take $e_1 = \partial_t,\ e_2 = t\partial_t + ku\partial_u,\ k \in \mathbb{R},\ e_3 = a(t)\partial_t + c(t)u\partial_u$, then $e_1, e_3] = 0$ gives $a, c \in \mathbb{R}$, so $e_3 = a\partial_t + cu\partial_u$. The relation $[e_2, e_3] = 0$ gives $a = 0$ so we have $e_3 = cu\partial_u,\ c \in \mathbb{R}$ and we then find $A_{3.2} = \langle\partial_t, t\partial_t, u\partial_u\rangle$. However, the vector fields $\partial_t, t\partial_t$ lead to $F = 0$, so this algebra is inadmissible. If we then take $e_1 = tu\partial_u,\ e_2 = -t\partial_t,\ e_3 = a(t)\partial_t + c(t)u\partial_u$, the relation $[e_1, e_3] = 0$ implies $a = 0$ so that $e_3 = c(t)u\partial_u$. Then $[e_2, e_3] = 0$ gives $-t\dot{c}u\partial_u = 0$ so that $c \in \mathbb{R}$ and it follows that we have $A_{3.2} = \langle tu\partial_u, -t\partial_t, u\partial_u\rangle$. However, the symmetry vector field $c(t)u\partial_u$ gives

$$uG_u + u_1G_{u_1} + u_2G_{u_2} = G + \frac{\dot{c}(t)}{c(t)}u$$

so that $c(t) = 1$ and $c(t) = t$ lead to a contradiction. Thus we have no admissible implementation of $A_{3.2}$ in this case.

$\underline{A_{3.3} : [e_2, e_3] = e_1,\ [e_1, e_2] = [e_1, e_3] = 0}$. As before, we implement $[e_1, e_2] = 0$ with either $e_1 = \partial_t,\ e_2 = u\partial_u$ or $e_1 = u\partial_u,\ e_2 = t\partial_t$. If $e_1 = \partial_t,\ e_2 = u\partial_u,\ e_3 = a(t)\partial_t + c(t)u\partial_u$, then we find as before that $e_3 = ae_1 + ce_2,\ a, c \in \mathbb{R}$ so that $\dim_{\mathbb{R}}\langle e_1, e_2, e_3\rangle = 2$, which is not possible. We then take $e_1 = u\partial_u,\ e_2 = t\partial_t,\ e_3 = a(t)\partial_t + c(t)u\partial_u$. The relation $[e_1, e_3] = 0$ is automatically satisfied, and $[e_2, e_3] = e_1$ gives $t\dot{a} = a,\ t\dot{c} = 1$ which give $a = kt,\ c = \ln|t| + l,\ k, l \in \mathbb{R}$ and we find that $\langle e_1, e_2, e_3\rangle = \langle u\partial_u, t\partial_t, \ln|t|u\partial_u\rangle$. However, the combination

the vector fields $u\partial_u$ and $\ln|t|u\partial_u$ cannot both be symmetries of our equation, for, as noted above, for a symmetry $c(t)u\partial_u$ we have

$$uG_u + u_1G_{u_1} + u_2G_{u_2} = G + \frac{\dot{c}(t)}{c(t)}u$$

so that $c(t) = 1$ and $c(t) = \ln|t|$ lead to a contradiction. Thus we have no admissible implementation of $A_{3.3}$ in this case.

For the algebras $A_{3.4} - A_{3.9}$ we require $[e_1, e_3] = e_1,\ [e_1, e_2] = 0,\ [e_2, e_3] \neq 0$. We may take either $e_1 = \partial_t,\ e_2 = u\partial_u$ or $e_1 = u\partial_u,\ e_2 = t\partial_t$ to implement $[e_1, e_2] = 0$. If $e_1 = \partial_t,\ e_2 = u\partial_u,\ e_3 = a(t)\partial_t + c(t)u\partial_u$, then we have $[e_2, e_3] = 0$, a contradiction. If $e_1 = u\partial_u,\ e_2 = t\partial_t$, then $[e_1, e_3] = 0$, again a contradiction. Thus we cannot find admissible implementations of these algebras.

We then have the following admissible direct sum extensions by three-dimensional solvable Lie algebras.

$$\langle 2x\partial_x, -x^2\partial_x, \partial_x\rangle \oplus \langle \partial_t, \partial_u, t\partial_t + \frac{u}{3}\partial_u\rangle.$$

The corresponding equation is given by

$$F = \frac{K}{u_1^3}, \quad G = -\frac{3K}{2}\frac{u_2^2}{u_1^4}$$

which leads to the Schwarzian KdV equation (up to a constant)

$$u_t = \frac{K}{u_x^3}u_{xxx} - \frac{3K}{2}\frac{u_{xx}^2}{u_x^4}.$$

6 Classification of equations invariant under low-dimensional solvable algebras

6.1 Equations with one-dimensional symmetry algebras

According to Theorem 2.3 we have three types of one-dimensional symmetry algebras

$$A_{1,1} : Q_1 = \partial_t, \quad A_{1,2} : Q_1 = \partial_x. \tag{6.1}$$

The corresponding invariant equations will have the form

$$A_{1,1} \quad : \quad u_t = F(x, u, u_1, u_2)u_3 + G(x, u, u_1, u_2), \tag{6.2a}$$

$$A_{1,2} \quad : \quad u_t = F(t, u, u_1, u_2)u_3 + G(t, u, u_1, u_2). \tag{6.2b}$$

Theorem 6.1 *There are two inequivalent classes of equations (2.6) invariant under one-parameter symmetry groups. Their representatives are given by (6.2).*

To obtain higher dimensional symmetry algebras we successively add further linearly independent vector fields to the one-dimensional ones and impose the commutation relations of all possible isomorphy classes of low-dimensional Lie algebras.

For two dimensional algebras we have the following result.

6.2 Equations with two-dimensional symmetry algebras

There are two isomorphy classes of two-dimensional Lie algebras, abelian and non-abelian satisfying the commutation relations $[Q_1, Q_2] = \kappa Q_2$, $\kappa = 0, 1$. We denote them by $A_{2,1}$ and $A_{2,2}$.

In order to obtain the two-dimensional realizations, to $A_{1,1}$ and $A_{1,2}$ we add a generic vector field (2.11) and impose the above commutators and then simplify further by equivalence transformations (2.4) respecting the canonical forms of 1-dimensional algebras. We find that there exist four and five inequivalent realizations of the algebra $A_{2.1}$ (in fact, we have already obtained them as 2-dimensional abelian algebras. See page 24) and $A_{2.2}$, respectively:

$$\begin{aligned}
A^1_{2.1} &= \langle \partial_t, \partial_u \rangle; \\
A^2_{2.1} &= \langle \partial_x, \partial_u \rangle; \\
A^3_{2.1} &= \langle \partial_x, t\partial_x \rangle; \\
A^4_{2.1} &= \langle \partial_x, u\partial_x \rangle,
\end{aligned}$$

$$\begin{aligned}
A^1_{2.2} &= \langle -t\partial_t - x\partial_x, \partial_t \rangle; \\
A^2_{2.2} &= \langle -t\partial_t - x\partial_x, \partial_x \rangle; \\
A^3_{2.2} &= \langle -x\partial_x - u\partial_u, \partial_x \rangle; \\
A^4_{2.2} &= \langle -x\partial_x, \partial_x \rangle; \\
A^5_{2.2} &= \langle -t\partial_t, \partial_t \rangle.
\end{aligned}$$

The corresponding forms of the functions F and G are given in Table 1. We note that the realizations $A^3_{2.1}$, $A^5_{2.2}$ can not be invariance algebras.

Theorem 6.2 *There exist nine classes of two-dimensional symmetry algebras admitted by equation (2.6). They are represented by the algebras $A^1_{2,1}, A^2_{2,1}, A^4_{2,1}$ and $A^1_{2,2}, \ldots, A^4_{2,2}$.*

6.3 Equations invariant under 3-dimensional solvable algebras

6.3.1 The Realizations of solvable decomposable algebras

There are two types of 3-dimensional decomposable Lie algebras,

$$A_{3,1} = 3A_1 = A_1 \oplus A_2 \oplus A_3$$

with commutation relations

$$[Q_i, Q_j] = 0, \qquad i, j = 1, 2, 3$$

and

$$A_{3,2} = A_{2,2} \oplus A_1$$

Table 1: Equations invariant under two-dimensional algebras

Algebra	F	G
$A^1_{2.1}$	$\tilde{F}(x, u_x, u_{xx})$	$\tilde{G}(x, u_x, u_{xx})$
$A^2_{2.1}$	$\tilde{F}(t, u_x, u_{xx})$	$\tilde{G}(t, u_x, u_{xx})$
$A^4_{2.1}$	$u_x^{-3}\tilde{F}(t, u, \omega)$, $\omega = u_x^{-3}u_{xx}$	$u_x\tilde{G}(t, u, \omega) - 3{u_x}^{-4}u_{xx}^2\tilde{F}(t, u, \omega)$
$A^1_{2.2}$	$x^2\tilde{F}(u, \omega_1, \omega_2)$	$x^{-1}\tilde{G}(u, \omega_1, \omega_2)$, $\omega_1 = xu_x$, $\omega_2 = x^2u_{xx}$
$A^2_{2.2}$	$t^2\tilde{F}(u, \omega_1, \omega_2)$	$t^{-1}\tilde{G}(u, \omega_1, \omega_2)$, $\omega_1 = tu_x$, $\omega_2 = t^2u_{xx}$
$A^3_{2.2}$	$u^3\tilde{F}(t, u_x, uu_{xx})$	$u\tilde{G}(t, u_x, uu_{xx})$
$A^4_{2.2}$	$u_x^{-3}\tilde{F}(t, u, \omega)$	$\tilde{G}(t, u, \omega)$, $\omega = u_x^{-2}u_{xx}$

with commutation relations

$$[Q_1, Q_2] = Q_2, \quad [Q_1, Q_3] = 0, \quad [Q_2, Q_3] = 0.$$

Applying the general strategy of introducing a new vector field Q_3 and invoking the above commutation relations we obtain the following realizations of decomposable algebras:

$$\begin{aligned}
A^1_{3.1} &= \langle \partial_t, \partial_u, \partial_x \rangle; \\
A^2_{3.1} &= \langle \partial_t, \partial_u, x\partial_u \rangle; \\
A^3_{3.1} &= \langle \partial_x, \partial_u, t\partial_x + f(t)\partial_u \rangle; \\
A^4_{3.1} &= \langle \partial_x, u\partial_x, f(t,u)\partial_x \rangle,
\end{aligned}$$

$$\begin{aligned}
A^1_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_t, \partial_u\rangle;\\
A^2_{3.2} &= \langle -t\partial_t - u\partial_u, \partial_t, xu\partial_u\rangle;\\
A^3_{3.2} &= \langle -t\partial_t - u\partial_u, \partial_u, t\partial_t + x\partial_x\rangle;\\
A^4_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_x, \partial_u\rangle;\\
A^5_{3.2} &= \langle -x\partial_x - u\partial_u, \partial_u, \partial_t\rangle;\\
A^6_{3.2} &= \langle -x\partial_x - u\partial_u, \partial_u, tx\partial_x\rangle;\\
A^7_{3.2} &= \langle -x\partial_x, \partial_x, \partial_t\rangle;\\
A^8_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_t, ut\partial_x\rangle;\\
A^9_{3.2} &= \langle -x\partial_x, \partial_x, \partial_u\rangle;\\
A^{10}_{3.2} &= \langle -x\partial_x - u\partial_u, \partial_x, u\partial_x\rangle;\\
A^{11}_{3.2} &= \langle -t\partial_t, \partial_t, \partial_x\rangle;\\
A^{12}_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_x, t\partial_x\rangle.
\end{aligned}$$

The corresponding forms of the functions F and G are given in Table 2. Note that $A^3_{3.1}$ and $A^{11}_{3.2}$, $A^{12}_{3.2}$ can not be invariance algebras.

Remark: The realization $A^4_{3.1}$, under the condition $f_{uu} \neq 0$, can be an invariance algebra of (2.6), with

$$\begin{aligned}
&F = u_x^{-3}\tilde{F}(t,u),\\
&G = u_x\tilde{G}(t,u) + u_x^{-2}u_{xx}f_{uu}^{-1}(f_t - f_{uuu}\tilde{F}) - 3u_x^{-4}u_{xx}^2\tilde{F}.
\end{aligned}$$

However, the coordinate change $x \to u, u \to x$ brings this equation to a linear one. The corresponding realization is then:

$$\langle \partial_u, x\partial_u, f(t,x)\partial_u\rangle, \quad f_{xx} \neq 0.$$

In the invariant equation $F = F(t,x)$, and $G = G(t,x,u_{xx})$ has to satisfy an equation linear in u_{xx}

$$f_t - f_{xxx}F(t,x) - f_{xx}G_{u_{xx}} = 0.$$

This implies that G should be linear in u_{xx}. We exclude such equations from the classification. For a group classification of third order linear evolution partial differential equations, the reader is referred to Ref. [4].

6.3.2 The realizations of non-decomposable solvable Lie algebras:

Every solvable Lie algebra of dimension three contains an Abelian ideal. Assuming that this ideal is $\{Q_1, Q_2\}$ that is already in canonical form $A^1_{2.1}$ and $A^2_{2.1}$, we add a third basis element Q_3. The commutation relations have the matrix form

$$\begin{pmatrix} [Q_1,Q_3] \\ [Q_2,Q_3] \end{pmatrix} = \mathbf{J}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \quad [Q_1,Q_2] = 0, \quad \mathbf{J} \in \mathbb{R}^{2\times 2}$$

where $\mathbf{J}$ can be taken in Jordan canonical form and normalized (multiplied by a nonzero constant). This is achieved by a change of basis in the ideal $\langle Q_1, Q_2\rangle$ and multiplying Q_3 by a real nonzero constant.

Table 2: Equations invariant under three-dimensional decomposable symmetry algebras

Algebra	F	G
$A^1_{3.1}$	$\tilde F(u_x, u_{xx})$	$\tilde G(u_x, u_{xx})$
$A^2_{3.1}$	$\tilde F(x, u_{xx})$	$\tilde G(x, u_{xx})$
$A^1_{3.2}$	$x^2\tilde F(\omega_1, \omega_2)$	$x^{-1}\tilde G(\omega_1, \omega_2),\ \omega_1 = xu_x,\ \omega_2 = x^2u_{xx}$
$A^2_{3.2}$	$u^{-1}e^{x\omega_1}\tilde F(x, \sigma)$	$e^{x\omega_1}[\tilde G(x, \sigma) - 3(\omega_1\omega_2 - \frac{2}{3}\omega_1^3)\tilde F(x, \sigma)]$, $\omega_1 = u^{-1}u_x,\ \omega_2 = u^{-1}u_{xx},\ \sigma = \omega_2 - \omega_1^2$
$A^3_{3.2}$	$t^{-1}x^3\tilde F(x^2\omega_1, \omega_1^{-3/2}\omega_2)$	$x^{-1}\tilde G(x^2\omega_1, \omega_1^{-3/2}\omega_2),\ \omega_1 = t^{-1}u_x,\ \omega_2 = t^{-1}u_{xx}$
$A^4_{3.2}$	$t^2\tilde F(tu_x, tu_{xx})$	$t^{-1}\tilde G(tu_x, tu_{xx})$,
$A^5_{3.2}$	$x^3\tilde F(u_x, xu_{xx})$	$x\tilde G(u_x, xu_{xx})$
$A^6_{3.2}$	$x^3\tilde F(t, \omega)$	$xt^{-1}u_x \ln\lvert u_x\rvert + xu_x\tilde G(t, \omega),\ \omega = xu_x^{-1}u_{xx}$
$A^7_{3.2}$	$u_x^{-3}\tilde F(u, \omega)$	$\tilde G(u, \omega),\ \omega = u_x^{-2}u_{xx}$
$A^8_{3.2}$	$t^{-1}u_x^{-3}\tilde F(u, \omega)$	$u_x\tilde G(u, \omega) - t^{-1}u - 3t^{-1}u_x^{-4}u_{xx}^2\tilde F(u, \omega),\ \omega = t^{-1}u_x^{-3}u_{xx}$
$A^9_{3.2}$	$u_x^{-3}\tilde F(t, \omega)$	$\tilde G(t, \omega),\ \omega = u_x^{-2}u_{xx}$
$A^{10}_{3.2}$	$u^3u_x^{-3}\tilde F(t, \omega)$	$uu_x\tilde G(u, \omega) - 3u^3u_x^{-4}u_{xx}^2\tilde F(t, \omega),\ \omega = uu_x^{-3}u_{xx}$

Table 3: Equations invariant under Weyl algebra

Algebra	F	G
$A^1_{3.3}$	$\tilde{F}(u_x, u_{xx})$	$x + \tilde{G}(u_x, u_{xx})$
$A^2_{3.3}$	$\tilde{F}(t, u_{xx})$	$-\frac{1}{2}u_x^2 + \tilde{G}(t, u_{xx})$

The set of three-dimensional solvable Lie algebras consists of the following two decomposable Lie algebras:

$$\begin{aligned} A_{3.1} &= A_1 \oplus A_1 \oplus A_1 = 3A_1; \\ A_{3.2} &= A_{2.2} \oplus A_1, \quad [Q_1, Q_2] = Q_2, \end{aligned}$$

and the following seven classes of non-decomposable Lie algebras:

$$\begin{aligned} A_{3.3} &: [Q_2, Q_3] = Q_1; \\ A_{3.4} &: [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = Q_1 + Q_2; \\ A_{3.5} &: [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = Q_2; \\ A_{3.6} &: [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = -Q_2; \\ A_{3.7} &: [Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = qQ_2 \;\; (0 < |q| < 1); \\ A_{3.8} &: [Q_1, Q_3] = -Q_2, \quad [Q_2, Q_3] = Q_1; \\ A_{3.9} &: [Q_1, Q_3] = qe_1 - Q_2, \quad [Q_2, Q_3] = Q_1 + qQ_2, \; (q > 0). \end{aligned}$$

We go through each possible isomorphy class. For the Weyl algebra $A_{3.3}$ we find

$$\begin{aligned} A^1_{3.3} &= \langle \partial_u, \partial_t, t\partial_u + \partial_x \rangle; \\ A^2_{3.3} &= \langle \partial_u, \partial_x, t\partial_x + x\partial_u \rangle. \end{aligned}$$

The forms of the functions F and G defining the corresponding equations are given in Table 3.

$$
\begin{aligned}
A_{3.4}^1 &= \langle \partial_u, \partial_t, t\partial_t + x\partial_x + (t+u)\partial_u \rangle; \\
&\quad F = \tilde{F}(u_x, xu_{xx}), \quad G = \tilde{G}(u_x, xu_{xx}) + \ln|x|;
\end{aligned}
$$

$$
\begin{aligned}
A_{3.4}^2 &= \langle \partial_u, \partial_t, t\partial_t + (t+u)\partial_u \rangle; \\
&\quad F = u_x^{-1}\tilde{F}(x, u_x^{-1}u_{xx}), \quad G = \tilde{G}(x, u_x^{-1}u_{xx}) + \ln|u_x|;
\end{aligned}
$$

$$
\begin{aligned}
A_{3.4}^3 &= \langle \partial_x, \partial_u, 2t\partial_t + (x+u)\partial_x + u\partial_u \rangle; \\
&\quad F = u_x^{-3}\exp{(u_x^{-1})}\tilde{F}(\omega_1, \omega_2), \quad G = u_x\exp{(-u_x^{-1})}u_{xx}\{-3\omega_2^2 u_x \tilde{F} + \tilde{G}(\omega_1, \omega_2)\}; \\
&\quad \omega_1 = 2u_x^{-1} - \ln|t|,\ \omega_2 = u_x^{-3}\exp(u_x^{-1})u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
A_{3.4}^4 &= \langle \partial_x, \partial_u, (x+u)\partial_x + u\partial_u \rangle; \\
&\quad F = u_x^{-3}\exp(3u_x^{-1})\tilde{F}(t, \omega), \quad G = u_x\exp(u_x^{-1})\{-3\omega^2\tilde{F}(t,\omega)u_x + \tilde{G}(t,\omega)\}, \\
&\quad \omega = u_x^{-3}\exp(u_x^{-1})u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
A_{3.5}^1 &= \langle \partial_t, \partial_u, t\partial_t + x\partial_x + u\partial_u \rangle; \\
&\quad F = x^2\tilde{F}(u_x, xu_{xx}), \quad G = \tilde{G}(u_x, xu_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A_{3.5}^2 &= \langle \partial_t, \partial_u, t\partial_t + u\partial_u \rangle; \\
&\quad F = u_x^{-1}\tilde{F}(x, u_x^{-1}u_{xx}), \quad G = \tilde{G}(x, u_x^{-1}u_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A_{3.5}^3 &= \langle \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u \rangle; \\
&\quad F = |t|^{\frac{1}{2}}\tilde{F}(u_x, tu_{xx}^2), \quad G = |t|^{-\frac{1}{2}}\tilde{G}(u_x, tu_{xx}^2);
\end{aligned}
$$

$$
\begin{aligned}
A_{3.6}^1 &= \langle \partial_t, \partial_u, t\partial_t + x\partial_x - u\partial_u \rangle; \\
&\quad F = x^2\tilde{F}(x^2u_x, x^3u_{xx}), \quad G = x^{-2}\tilde{G}(x^2u_x, x^3u_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A^2_{3.6} &= \langle \partial_t, \partial_u, t\partial_t - u\partial_u\rangle;\\
&F = u_x\tilde{F}(x, u_x^{-1}u_{xx}), \quad G = u_x^2\tilde{G}(x, u_x^{-1}u_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A^3_{3.6} &= \langle \partial_x, \partial_u, t\partial_t + x\partial_x - u\partial_u\rangle;\\
&F = t^2\tilde{F}(t^2u_x, t^3u_{xx}), \quad G = t^{-2}\tilde{G}(t^2u_x, t^3u_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A^4_{3.6} &= \langle \partial_x, \partial_u, x\partial_x - u\partial_u\rangle;\\
&F = t^{-1}\tilde{F}(tu_x, tu_{xx}), \quad G = t^{-2}\tilde{G}(tu_x, tu_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A^1_{3.7} &= \langle \partial_u, \partial_t, qt\partial_t + x\partial_x + u\partial_u\rangle \ (q \neq 0, \pm 1);\\
&F = |x|^{3-q}\tilde{F}(u_x, xu_{xx}), \quad G = |x|^{1-q}\tilde{G}(u_x, xu_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A^2_{3.7} &= \langle \partial_u, \partial_t, qt\partial_t + u\partial_u\rangle \ (q \neq 0, \pm 1);\\
&F = |u_x|^{-q}\tilde{F}(x, u_x^{-1}u_{xx}), \quad G = |u_x|^{1-q}\tilde{G}(x, u_x^{-1}u_{xx});
\end{aligned}
$$

$$
\begin{aligned}
A^3_{3.7} &= \langle \partial_x, \partial_u, t\partial_t + x\partial_x + qu\partial_u\rangle \ (0 < |q| < 1);\\
&F = t^2\tilde{F}(\omega_1, \omega_2), \quad G = |t|^{q-1}\tilde{G}(\omega_1, \omega_2);\\
&\omega_1 = |t|^{1-q}u_x, \ \omega_2 = |t|^{2-q}u_{xx}
\end{aligned}
$$

$$
\begin{aligned}
A^4_{3.7} &= \langle \partial_x, \partial_u, x\partial_x + qu\partial_u\rangle \ (0 < |q| < 1);\\
&F = |u_x|^{\frac{3}{q-1}}\tilde{F}(t, \omega), \quad G = |u_x|^{\frac{q}{q-1}}\tilde{G}(t, \omega), \quad \omega = |u_x|^{(2-q)/(q-1)}u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
A^1_{3.8} &= \langle \partial_x, \partial_u, \partial_t + u\partial_x - x\partial_u\rangle;\\
&F = (1+u_x^2)^{-3/2}\tilde{F}(\omega_1, \omega_2), \quad G = \sqrt{1+u_x^2}\{-3\omega_2^2\tilde{F}(\omega_1, \omega_2)u_x + \tilde{G}(\omega_1, \omega_2)\};\\
&\omega_1 = t + \arctan u_x, \quad \omega_2 = (1+u_x^2)^{-3/2}u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
A^2_{3.8} &= \langle \partial_x, \partial_u, u\partial_x - x\partial_u\rangle;\\
&F = (1+u_x^2)^{-3/2}\tilde{F}(t, \omega), \quad G = \sqrt{1+u_x^2}\{-3\omega^2u_x\tilde{F}(t, \omega) + \tilde{G}(t, \omega)\};\\
&\omega = (1+u_x^2)^{-3/2}u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
A^1_{3.9} \;=\;& \langle \partial_x, \partial_u, \partial_t + (u+qx)\partial_x + (qu-x)\partial_u\rangle \ (q>0);\\
& F = (1+u_x^2)^{-3/2}\exp(-3q\arctan u_x)\tilde{F}(\omega_1,\omega_2),\\
& G = \sqrt{1+u_x^2}\exp(-q\arctan u_x)\{-3\omega_2^2 u_x\tilde{F}(\omega_1,\omega_2)+\tilde{G}(\omega_1,\omega_2)\};\\
& \omega_1 = t+\arctan u_x,\quad \omega_2 = (1+u_x^2)^{-3/2}\exp(-q\arctan u_x)u_{xx};
\end{aligned}
$$

$$
\begin{aligned}
A^2_{3.9} \;=\;& \langle \partial_x, \partial_u, (u+qx)\partial_x + (qu-x)\partial_u\rangle \ (q>0);\\
& F = (1+u_x^2)^{-3/2}\exp(-3q\arctan u_x)\tilde{F}(t,\omega),\\
& G = \sqrt{1+u_x^2}\exp(-q\arctan u_x)\{-3\omega^2 u_x\tilde{F}(t,\omega)+\tilde{G}(t,\omega)\};\\
& \omega = (1+u_x^2)^{-3/2}\exp(-q\arctan u_x)u_{xx}.
\end{aligned}
$$

There are two inequivalent realizations of the nilpotent algebra $A_{3.3}$

$$
\begin{aligned}
A^1_{3.3} &= \langle \partial_u, \partial_t, t\partial_u + \partial_x\rangle;\\
A^2_{3.3} &= \langle \partial_u, \partial_x, t\partial_x + x\partial_u\rangle.
\end{aligned}
$$

The forms of the functions F and G defining the corresponding nonlinear equations are given in Table 3.

Theorem 6.3 *There are thirty-five inequivalent three-dimensional solvable symmetry algebras admitted by equation* (2.6).

6.4 Equations invariant under four-dimensional Algebras

For $\dim L = 4$, we proceed exactly in the same manner as before. We start from the already standardized three-dimensional algebras, and add a further linearly independent element Q_4, and require that they satisfy the commutators of the four-dimensional non-isomorphic representative Lie algebras. We distinguish between two types of algebras, decomposable and non-decomposable:

A. Decomposable Algebras:

The set of inequivalent abstract four-dimensional Lie algebras contains ten real decomposable Lie algebras represented by

$$
4A_1 = A_{3.1}\oplus A_1,\ A_{2.2}\oplus 2A_1 = A_{2.2}\oplus A_{2.1},\ A_{2.2}\oplus A_{2.2} = 2A_{2.2}, A_{3.i}\oplus A_1\ (i=3,4,\ldots,9)
$$

Below, omitting details we give the symmetry algebras and the associated invariant equations:

$$2A_{2.2}^1 = A_{3.2}^1 \oplus \langle -u\partial_u + kx\partial_x\rangle \quad (k \neq 0);$$
$$F = \tilde{F}(\omega)x^2|xu_x|^{-k}, \quad G = u_x|xu_x|^{-k}\tilde{G}(\omega), \quad \tilde{F}(\omega) \neq 0,$$
$$\omega = xu_x^{-1}u_{xx};$$

$$2A_{2.2}^2 = A_{3.2}^2 \oplus \langle x\partial_x\rangle;$$
$$F = \tilde{F}(\omega)x^3u^{-1}\exp(x\omega_1),$$
$$G = \exp(x\omega_1)\{(-3u^{-2}u_xu_{xx} + 2u^{-3}u_x^3)\tilde{F}(\omega) + \tilde{G}(\omega)\},$$
$$\omega_1 = u^{-1}u_x, \quad \omega = x^2(u^{-1}u_{xx} - u^{-2}u_x^2);$$

$$2A_{2.2}^3 = A_{3.2}^4 \oplus \langle -u\partial_u + kt\partial_t\rangle \quad (k \neq 0, 1);$$
$$F = \tilde{F}(\omega)|\omega_1|^{\frac{3k}{1-k}}, \quad G = t^{-1}|\omega_1|^{\frac{1}{1-k}}\tilde{G}(\omega),$$
$$\omega_1 = tu_x, \ \omega_2 = t^2u_{xx}, \ \omega = \omega_1^{\frac{1-2k}{k-1}}\omega_2, \quad \tilde{F}(\omega) \neq 0;$$

$$2A_{2.2}^4 = A_{3.2}^4 \oplus \langle -u\partial_u + t\partial_x\rangle;$$
$$F = \tilde{F}(\omega)t^2, \quad G = tu_x[\ln|tu_x| + \tilde{G}(\omega)], \quad \tilde{F}(\omega) \neq 0, \quad \omega = tu_x^{-1}u_{xx}.$$

B. Nondecomposable algebras:

The set of inequivalent abstract four-dimensional Lie algebras contains ten real non-decomposable Lie algebras $A_{4,i} = \langle Q_1, Q_2, Q_3, Q_4\rangle$ $(i = 1, \ldots, 10)$. They are all solvable and therefore can be written as semidirect sums of a one-dimensional Lie algebra $\langle Q_4\rangle$ and a three-dimensional ideal $N = \langle Q_1, Q_2, Q_3\rangle$. For $A_{4,i}$ $(i = 1, \ldots, 6)$, N is abelian, for $A_{4,7}, A_{4,8}, A_{4,9}$ it is of type $A_{3,3}$ (nilpotent), and for $A_{4,10}$ it is of the type $A_{3,5}$.

The non-decomposable solvable Lie algebras are represented by

$A_{4.1}$: $[Q_2, Q_4] = Q_1, \quad [Q_3, Q_4] = Q_2;$

$A_{4.2}$: $[Q_1, Q_4] = qQ_1, \quad [Q_2, Q_4] = Q_2, \quad [Q_3, Q_4] = Q_2 + Q_3, \quad q \neq 0;$

$A_{4.3}$: $[Q_1, Q_4] = Q_1, \quad [Q_3, Q_4] = Q_2;$

$A_{4.4}$: $[Q_1, Q_4] = Q_1, \quad [Q_2, Q_4] = Q_1 + Q_2, \quad [Q_3, Q_4] = Q_2 + Q_3;$

$A_{4.5}$: $[Q_1, Q_4] = Q_1, \quad [Q_2, Q_4] = qQ_2, \quad [Q_3, Q_4] = pQ_3, \quad -1 \leq p \leq q \leq 1, \quad p \cdot q \neq 0;$

$A_{4.6}$: $[Q_1, Q_4] = qQ_1, \quad [Q_2, Q_4] = pQ_2 - Q_3, \quad [Q_3, Q_4] = Q_2 + pQ_3, \quad q \neq 0, \quad p \geq 0;$

$A_{4.7}$: $[Q_2, Q_3] = Q_1, \quad [Q_1, Q_4] = 2Q_1, \quad [Q_2, Q_4] = Q_2, \quad [Q_3, Q_4] = Q_2 + Q_3;$

$A_{4.8}$: $[Q_2, Q_3] = Q_1, \quad [Q_1, Q_4] = (1 + q)Q_1, \quad [Q_2, Q_4] = Q_2, \quad [Q_3, Q_4] = qe_3, \quad |q| \leq 1;$

$A_{4.9}$: $[Q_2, Q_3] = Q_1, \quad [Q_1, Q_4] = 2qQ_1, \quad [Q_2, Q_4] = qQ_2 - Q_3, \quad [Q_3, Q_4] = Q_2 + qQ_3, \ q \geq 0;$

$A_{4.10}$: $[Q_1, Q_3] = Q_1, \quad [Q_2, Q_3] = Q_2, \quad [Q_1, Q_4] = -Q_2, \quad [Q_2, Q_4] = Q_1.$

As the algebra $A_{4,8}$ contains the symmetry algebra of the KdV equation we shall give a detailed analysis of the possible inequivalent realizations of $A_{4,8}$ below.

$A = A_{4.8}$

This algebra has the following commutation scheme:

$$
\begin{aligned}
&[e_1, e_2] = 0, \quad [e_1, e_3] = 0, \quad [e_1, e_4] = (1+q)e_1, \\
&\qquad\qquad\qquad [e_2, e_3] = e_1, \quad [e_2, e_4] = e_2, \quad |q| \leq 1, \\
&\qquad\qquad\qquad\qquad\qquad\qquad [e_3, e_4] = qe_3.
\end{aligned}
$$

To begin the classification of the inequivalent realizations, we note that $\langle e_1, e_2 \rangle$ is an abelian subalgebra, and so we know that there are the following admissible forms for $\langle e_1, e_2 \rangle$ (note that we now have an **ordered** basis):

$$\langle \partial_t, \partial_x \rangle, \quad \langle \partial_x, \partial_t \rangle, \quad \langle \partial_x, u\partial_u \rangle.$$

If we take $\langle \partial_t, \partial_x \rangle$ first, we note that with $e_3 = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$, the relation $[e_2, e_3] = e_1$ leads to

$$b_x\partial_x + c_x\partial_u = \partial_t,$$

which is obviously impossible. So we only have two possibilities for $\langle e_1, e_2 \rangle$.

$\underline{\langle e_1, e_2 \rangle = \langle \partial_x, u\partial_x \rangle}$. Note that we have $\mathcal{E}(e_1) \cap \mathcal{E}(e_2) : t' = T(t),\ \dot{T} \neq 0; x' = x + Y(t,u); u' = u$. Then we take $e_3 = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$. Then $[e_1, e_3] = 0$ gives $e_3 = a(t)\partial_t + b(t,u)\partial_x + c(t,u)\partial_u$. The relation $[e, e_3] = e_1$ yields $c = -1$ and this gives us $e_3 = a(t)\partial_t b(t,u)\partial_x - \partial_u$. Then transforming e_3 by an equivalence transformation of the type given above, we find

$$e_3' = a(t)\dot{T}(t)\partial_{t'} + [b(t,u) - Y_u + a(t)Y_t]\partial_{x'} - \partial_{u'}.$$

We may clearly choose $Y(t,u)$ so that $b(t,u) - Y_u + a(t)Y_t = 0$ so that we find $e_3' = a(t)\dot{T}(t)\partial_{t'} - \partial_{u'}$. If $a \neq 0$ we choose T so that $a(t)\dot{T}(t) = 1$, giving $e_3' = \partial_{t'} - \partial_{u'}$. If, on the other hand, $a = 0$ then we have $e_3' = -\partial_{u'}$. Thus, we have the two possibilities

$$\langle e_1, e_2, e_3 \rangle = \langle \partial_x, u\partial_x, -\partial_u \rangle, \quad \langle e_1, e_2, e_3 \rangle = \langle \partial_x, u\partial_x, \partial_t - \partial_u \rangle.$$

If we take $\langle e_1, e_2, e_3 \rangle = \langle \partial_x, u\partial_x, -\partial_u \rangle$, we find, with $e_4 = a(t)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u$, that $b_x = (1+q),\ \ c_x = 0$ since $[e_1, e_4] = (1+q)e_1$, and $ub_x - c = u$ since $[e_2, e_4] = e_2$. Furthermore, $[e_3, e_4] = qe_3$ yields $b_u = 0,\ \ c_u = 1$. From these equations we find $b = (1+q)x + \beta(t),\ \ c = qu$ where $\alpha \in \mathbb{R}$.

The equivalence group is $\mathcal{E}(e_1) \cap \mathcal{E}(e_2) \cap \mathcal{E}(e_3) : t' = T(t),\ \ \dot{T} \neq 0;\ \ x' = x + Y(t),\ \ u' = u$. Applying such an equivalence relation we find that e_4 is mapped to

$$
\begin{aligned}
e_4' &= a(t)\dot{T}(t)\partial_{t'} + [(1+q)x + \beta(t) + a(t)Y'(t)]\partial_{x'} + qu\partial_{u'} \\
&= a(t)\dot{T}(t)\partial_{t'} + [(1+q)x' + \beta(t) + a(t)Y'(t) - (1+q)Y(t)]\partial_{x'} + qu'\partial_{u'}.
\end{aligned}
$$

From this, we see that for $a(t) \neq 0$ we may choose $T(t),\ Y(t)$ so that $a(t)\dot{T}(t) = T(t)$ and $\beta(t) + a(t)Y'(t) - (1+q)Y(t) = 0$, giving

$$e_4' = t'\partial_{t'} + (1+q)x'\partial_{x'} + qu'\partial_{u'}.$$

If $a(t)=0$ then $e_4'=[(1+q)x'+\beta(t)-(1+q)Y(t)]\partial_{x'}+qu'\partial_{u'}$. If $q\neq -1$ we choose $Y(t)$ so that $\beta(t)-(1+q)Y(t)=0$ and we obtain $e_4'=(1+q)x'\partial_{x'}+qu'\partial_{u'}$. If $q=-1$ and $a(t)=0$ we have $e_4'=\beta(t)\partial_{x'}-u'\partial_{u'}$. If $\dot{\beta}(t)=0$ then we have $\beta\partial_{x'}-u'\partial_{u'}$ with $\beta\in\mathbb{R}$. If $\dot{\beta}(t)\neq 0$ we may take $\beta(t)=T(t)$ and then $e_4'=t'\partial_{x'}-u'\partial_{u'}$. Consequently, we find the following canonical forms for $A_{4.8}$:

$$\begin{aligned}
\langle e_1,e_2,e_3,e_4\rangle &= \langle \partial_x, u\partial_x, -\partial_u, t\partial_t+(1+q)x\partial_x+qu\partial_u\rangle,\\
\langle e_1,e_2,e_3,e_4\rangle &= \langle \partial_x, u\partial_x, -\partial_u, (1+q)x\partial_x+qu\partial_u\rangle,\\
\langle e_1,e_2,e_3,e_4\rangle &= \langle \partial_x, u\partial_x, -\partial_u, t\partial_x-u\partial_u\rangle.
\end{aligned}$$

Now we suppose that $\langle e_1,e_2,e_3\rangle=\langle\partial_x,u\partial_x,\partial_t-\partial_u\rangle$. The equivalence group is $\mathcal{E}(e_1)\cap\mathcal{E}(e_2)\cap\mathcal{E}(e_3): t'=t+k; x'=x+Y(t+u); u'=u$. Putting $e_4=a(t)\partial_t+b(t,x,u)\partial_x+c(t,x,u)\partial_u$ we have from $[e_1,e_4]=(1+q)e_1$ that $b_x=(1+q)$, $c_x=0$; from $[e_2,e_4]=e_2$ we have $ub_x-c=u$; and from $[e_3,e_4]=qe_3$ we have $\dot{a}(t)=q$, $b_t-b_u=0$, $c_t-c_u=-q$. From these equations we find that $a(t)=qt+\alpha$, $b=(1+q)x+\beta(t+u)$, $c=qu$ where $\alpha\in\mathbb{R}$ and $\beta(t+u)$ is a smooth function. Thus,

$$e_4=[qt+\alpha]\partial_t+[(1+q)x+\beta(t+u)]\partial_x+qu\partial_u.$$

Applying an equivalence transformation, we find that e_4 is mapped to

$$\begin{aligned}
e_4'=&[qt+\alpha]\partial_{t'}+[(1+q)x+\beta(t+u)+q(t+u)Y'(t+u)+\alpha Y'(t+u)]\partial_{x'}+qu\partial_{u'}\\
=&[qt'+\alpha-qk]\partial_{t'}+[(1+q)x'+\beta(t+u)+q(t+u)Y'(t+u)\\
&+\alpha Y'(t+u)-(1+q)Y(t+u)]\partial_{x'}+qu'\partial_{u'}.
\end{aligned}$$

From this we see that we may always choose $Y(t+u)$ so that $\beta(t+u)+q(t+u)Y'(t+u)+\alpha Y'(t+u)-(1+q)Y(t+u)=0$, so we always have the canonical form

$$e_4'=[qt'+\alpha-qk]\partial_{t'}+(1+q)x'\partial_{x'}+qu'\partial_{u'}.$$

When $q\neq 0$ we may choose k so that $\alpha-qk=0$ and we then have $e_4'=qt'\partial_{t'}+(1+q)x'\partial_{x'}+qu'\partial_{u'}$. If $q=0$ then we have $e_4'=\alpha\partial_{t'}+x'\partial_{x'}$ with $\alpha\in\mathbb{R}$. Consequently, we have the following canonical forms for $A_{4.8}$:

$$\begin{aligned}
\langle e_1,e_2,e_3,e_4\rangle &= \langle \partial_x, u\partial_x, \partial_t-\partial_u, qt\partial_t+(1+q)x\partial_x+qu\partial_u\rangle,\\
\langle e_1,e_2,e_3,e_4\rangle &= \langle \partial_x, u\partial_x, \partial_t-\partial_u, \alpha\partial_t+x\partial_x\rangle,\quad \alpha\in\mathbb{R}.
\end{aligned}$$

$\underline{\langle e_1,e_2\rangle=\langle\partial_x,\partial_t\rangle}$. The equivalence algebra is $\mathcal{E}(e_1)\cap\mathcal{E}(e_2): t'=t+k;\ x'=x+Y(u);\ u'=U(u),\ U'(u)\neq 0$. With $e_3=a(t)\partial_t+b(t,x,u)\partial_x+c(t,x,u)\partial_u$ we find that $[e_1,e_3]=0$ gives $b_x=c_x=0$ and that $[e_2,e_3]=e_1$ yields $\dot{a}=0,\ b_t=1,\ c_t=0$ from which we deduce that $a\in\mathbb{R},\ b=t+\beta(u),\ c=c(u)$. So $e_3=a\partial_t+[t+\beta(u)]\partial_x+c(u)\partial_u$. Applying an equivalence transformation, we find that e_3 is mapped to $e_3'=a\partial_{t'}+[t'+\beta(u)-k+c(u)Y'(u)]\partial_{x'}+c(u)U'(u)\partial_{u'}$. If $c(u)\neq 0$ we choose $U(u)$ and $Y(u)$ so that $c(u)U'(u)=-1,\ c(u)Y(u)+\beta(u)-k=0$ and we then have $e_3'=a\partial+t'\partial_{x'}-\partial_{u'}$ (the

reason for this choice of $U(u)$ is that this symmetry appears for the KdV-equation $u_t = u_3 + uu_1$). If $c(u) = 0$ then we have $e_3' = a\partial_{t'} + [t' + \beta(u) - k]\partial_{x'}$. Then if $\beta'(u) = 0$ we choose $k = \beta$ so that $e_3' = a\partial_{t'} + t'\partial_{x'}$; if $\beta'(u) \neq 0$ then we take $U(u) = \beta(u) - k$ and we have $e_3' = a\partial_{t'} + (t' + u')\partial_{u'}$. So we have the following possibilities:

$$\begin{aligned}\langle e_1, e_2, e_3\rangle =& \langle \partial_x, \partial_t, t\partial_x - \partial_u\rangle\\ \langle e_1, e_2, e_3\rangle =& \langle \partial_x, \partial_t, (t+u)\partial_x\rangle.\end{aligned}$$

The realization $\langle e_1, e_2, e_3\rangle = \langle \partial_x, \partial_t, t\partial_x\rangle$ is not admissible since $\partial_x, t\partial_x$ gives us $u_1 = 0$ which is a contradiction.

$\underline{\langle e_1, e_2, e_3\rangle = \langle \partial_x, \partial_t, t\partial_x - \partial_u\rangle}$. The equivalence group is $\mathcal{E}(e_1) \cap \mathcal{E}(e_2) \cap \mathcal{E}(e_3) : t' = t + k; x' = x - ku + l; u' = u + m$ with $k, l, m \in \mathbb{R}$. Put $e_4 = a(t)\partial_t + b(t, x, u)\partial_x + c(t, x, u)\partial_u$. The relations $[e_1, e_4] = (1 + q)e_1,\ [e_2, e_4] = e_2,\ [e_3, e_4] = qe_3$ yield $c_t = c_x = 0,\ b_t = 0, b_x = 1 + q,\ \dot{a}(t) = 1$, from which we find that $a(t) = t + \alpha,\ b = (1 + q)x - \alpha u + \beta,\ c = qu + \gamma$ with $\alpha, \beta, \gamma \in \mathbb{R}$. Under an equivalence transformation e_4 is mapped to

$$\begin{aligned}e_4' =& (t + \alpha)\partial_{t'} + [(1 + q)x - \alpha u + \beta - k(qu + \gamma)]\partial_{x'} + [qu + \gamma]\partial_{u'}\\ =& (t' + \alpha - k)\partial_{t'} + [(1 + q)x' + (k - \alpha)u + \beta - k\gamma - l(1 + q)]\partial_{x'} + [qu' + \gamma - mq]\partial_{u'}.\end{aligned}$$

We may always choose $k = \alpha$ and this gives

$$e_4' = t'\partial_{t'} + [(1 + q)x' + \beta - \alpha\gamma - l(1 + q)]\partial_{x'} + [qu' + \gamma - mq]\partial_{u'}.$$

If $q \neq -1, 0$ we choose l, m so that $\beta - \alpha\gamma - l(1 + q) = 0,\ \gamma - mq = 0$ and then

$$e_4' = t'\partial_{t'} + (1 + q)x'\partial_{x'} + qu'\partial_{u'}.$$

If $q = -1$ we choose m so that $\gamma + m = 0$ and then

$$e_4' = t'\partial_{t'} + (\beta - m\alpha)\partial_{x'} - u'\partial_{u'}.$$

If $q = 0$ we choose l so that $\beta - \alpha\gamma - l = 0$ and we find

$$e_4' = t'\partial_{t'} + x'\partial_{x'} + \gamma\partial_{u'}.$$

From this we find the following canonical forms for $A_{4.8}$:

$$\begin{aligned}\langle e_1, e_2, e_3, e_4\rangle =& \langle \partial_x, \partial_t, t\partial_x - \partial_u, t\partial_t + (1 + q)x\partial_x + qu\partial_u\rangle\\ \langle e_1, e_2, e_3, e_4\rangle =& \langle \partial_x, \partial_t, t\partial_x - \partial_u, t\partial_t + x\partial_x + \kappa\partial_u\rangle, \kappa \in \mathbb{R}.\end{aligned}$$

$\underline{\langle e_1, e_2, e_3\rangle = \langle \partial_x, \partial_t, (t + u)\partial_x\rangle}$. The equivalence group is $\mathcal{E}(e_1) \cap \mathcal{E}(e_2) \cap \mathcal{E}(e_3) : t' = t + k; x' = x + Y(u); u' = u - k,\ k \in \mathbb{R}$. With $e_4 = a(t)\partial_t + b(t, x, u)\partial_x + c(t, x, u)\partial_u$, the relations $[e_1, e_4] = (1 + q)e_1,\ [e_2, e_4] = e_2$ yield $c_t = c_x = b_t = 0, b_x = 1 + q,\ \dot{a}(t) = 1$, from which we find that $a(t) = t + \alpha,\ b = (1 + q)x + \beta(u),\ c = c(u)$ with $\alpha \in \mathbb{R}$. Then $[e_3, e_4] = qe_3$ gives $c(u) = u - \alpha$ so that

$$e_4 = (t + \alpha)\partial_t + [(1 + q)x + \beta(u)]\partial_x + (u - \alpha)\partial_u.$$

Under an equivalence transformation e_4 is mapped to

$$\begin{aligned} e_4' =& (t+\alpha)\partial_{t'} + [(1+q)x + \beta(u)]\partial_{x'} + (u-\alpha)\partial_{u'} \\ =& (t'+\alpha-k)\partial_{t'} + [(1+q)x' + \beta(u) + (u-\alpha)Y'(u) - (1+q)Y(u)]\partial_{x'} + (u'+k-\alpha)\partial_{u'}. \end{aligned}$$

We may always choose $k=\alpha$ and $Y(u)$ so that $\beta(u)+(u-\alpha)Y'(u)-(1+q)Y(u)=0$ so that we find

$$e_4' = t'\partial_{t'} + (1+q)x'\partial_{x'} + u'\partial_{u'}.$$

We then have the following realization of $A_{4.8}$:

$$\langle e_1, e_2, e_3, e_4\rangle = \langle \partial_x, \partial_t, (t+u)\partial_x, t\partial_t + (1+q)x\partial_x + u\partial_u\rangle.$$

We then have the following realizations of $A_{4.8}$:

$$\begin{aligned} &\langle \partial_x, u\partial_x, -\partial_u, t\partial_t + (1+q)x\partial_x + qu\partial_u\rangle, \\ &\langle \partial_x, u\partial_x, -\partial_u, (1+q)x\partial_x + qu\partial_u\rangle, \\ &\langle \partial_x, u\partial_x, -\partial_u, t\partial_x - u\partial_u\rangle \\ &\langle \partial_x, u\partial_x, \partial_t - \partial_u, qt\partial_t + (1+q)x\partial_x + qu\partial_u\rangle, \\ &\langle \partial_x, u\partial_x, \partial_t - \partial_u, \alpha\partial_t + x\partial_x\rangle, \quad \alpha \in \mathbb{R} \\ &\langle \partial_x, \partial_t, t\partial_x - \partial_u, t\partial_t + (1+q)x\partial_x + qu\partial_u\rangle \\ &\langle \partial_x, \partial_t, t\partial_x - \partial_u, t\partial_t + x\partial_x + \kappa\partial_u\rangle, \kappa \in \mathbb{R} \\ &\langle \partial_x, \partial_t, (t+u)\partial_u, t\partial_t + (1+q)x\partial_x + u\partial_u\rangle. \end{aligned}$$

The corresponding admissible invariant equations:
$\underline{\langle \partial_x, u\partial_x, -\partial_u, t\partial_t + (1+q)x\partial_x + qu\partial_u\rangle}$. In this case, the operators $\partial_x, u\partial_x$ give us $F_x = G_x = 0$ and

$$\begin{aligned} u_1F_{u_1} + 3u_2F_{u_2} + 3F = 0 \\ u_1^2G_{u_1} + 3u_1u_2G_{u_2} + 3u_2^2F = u_1G, \end{aligned}$$

which integrate to give

$$F = u_1^{-3}f(t,u,\sigma), \quad G = u_1g(t,u,\sigma) - 3u_1^2\sigma^2 f(t,u,\sigma),$$

where $\sigma = u_1^{-3}u_2$. Thus we have with ∂_u as a symmetry that

$$F = u_1^{-3}f(t,\sigma), \quad G = u_1g(t,\sigma) - 3u_1^2\sigma^2 f(t,\sigma).$$

It is useful to change coordinates: $(t,x,u,u_1,u_2) \to (t,x,u,u_1,\sigma)$. The operator $t\partial_t + (1+q)x\partial_x + qu\partial_u$ then gives us the system

$$\begin{aligned} t\hat{F}_t = u_1\hat{F}_{u_1} + (q-1)\sigma\hat{F}_\sigma + (3q+2)\hat{F} \\ t\hat{G}_t = u_1\hat{G}_{u_1} + (q-1)\sigma\hat{G}_\sigma + (q-1)\hat{G}, \end{aligned}$$

where $\hat{F}(t, x, u, u_1, \sigma) = F(t, x, u, u_1, u_2)$ and similarly for G. These equations then give

$$tf_t = (q-1)\sigma f_\sigma + (3q-1)f$$

from the equation for F and

$$\begin{aligned} tg_t &= (q-1)\sigma g_\sigma + qg \\ tf_t &= (q-1)\sigma f_\sigma + (q+1)f \end{aligned}$$

from the equation for G. Clearly, we must have $3q - 1 = q + 1$ for compatibility, which gives $q = 1$. Thus, we have an implementation of this realization of $A_{4.8}$ only for $q = 1$. Then our equations for f, g become

$$tf_t = 2f, \quad tg_t = g,$$

from which we conclude that

$$f = t^2\phi(\sigma), \quad g = t\psi(\sigma),$$

and then we have

$$F = t^2 u_1^{-3}\phi(\sigma), \quad G = tu_1\psi(\sigma) - 3t^2u_1^2\sigma^2\phi(\sigma).$$

$\underline{\langle \partial_x, u\partial_x, -\partial_u, (1+q)x\partial_x + qu\partial_u \rangle}$. In this case, the operators ∂_x, ∂_u give us $F_x = G_x = F_u = G_u = 0$ and $u\partial_x$ gives us

$$F = u_1^{-3}f(t,\sigma), \quad G = u_1 g(t,\sigma) - 3u_1^2\sigma^2 f(t,\sigma).$$

The operator $(1+q)x\partial_x + qu\partial_u$ then gives us the equations

$$\begin{aligned} u_1F_{u_1} + (q+2)u_2F_{u_2} + 3(q+1)F &= 0 \\ u_1G_{u_1} + (q+2)u_2G_{u_2} + qG &= 0, \end{aligned}$$

which is, on changing variables to (t, x, u, u_1, σ), and putting $\hat{F}(t, x, u, u_1, \sigma) = F(t, x, u, u_1, u_2)$,

$$\begin{aligned} u_1\hat{F}_{u_1} + (q-1)\sigma\hat{F}_{u_2} + 3(q+1)\hat{F} &= 0 \\ u_1\hat{G}_{u_1} + (q-2)\sigma\hat{G}_{u_2} + qG &= 0, \end{aligned}$$

and these equations then give us

$$(q-1)\sigma f_\sigma + 3qf = 0, \quad (q-1)\sigma g_\sigma + (q+1)g = 0.$$

Clearly, $q = 1$ is inadmissible for then we would have $f = 0$ and then $F = 0$. So for $-1 \le q < 1$ we have

$$f(t,\sigma) = \sigma^{3q/(1-q)}\phi(t), \quad g(t,\sigma) = \sigma^{(1+q)/(1-q)}\psi(t).$$

From this we have

$$F = u_1^{-3}\sigma^{3q/(1-q)}\phi(t), \quad G = u_1\sigma^{(1+q)/(1-q)}\psi(t) - 3u_1^2\sigma^2\sigma^{q/(1-q)}\phi(t).$$

$\underline{\langle \partial_x, u\partial_x, -\partial_u, t\partial_x - u\partial_u \rangle}$. In this case we have, from the operator $t\partial_x - u\partial_u$, the equations

$$\begin{aligned} u_1 F_{u_1} + u_2 F_{u_2} &= 0 \\ u_1 G_{u_1} + u_2 G_{u_2} &= u_1 + G. \end{aligned}$$

From these we find, proceeding as before,

$$F = u_1^{-3}\sigma^{-3/2}\phi(t), \quad G = u_1[\psi(t) - \frac{1}{2}\ln\sigma] - 3u_1^2\sigma^{1/2}\phi(t).$$

$\underline{\langle \partial_x, u\partial_x, \partial_t - \partial_u, qt\partial_t + (1+q)x\partial_x + qu\partial_u \rangle}$. In this case we have, from the operators $\partial_x, u\partial_x$

$$F = u_1^{-3} f(t,u,\sigma), \quad G = u_1 g(t,u,\sigma) - 3u_1^2\sigma^2 f(t,u,\sigma).$$

Then invariance under the operator $\partial_t - \partial_u$ (we find $F_t - F_u = G_t - G_u = 0$) gives us

$$F = u_1^{-3} f(\tau,\sigma), \quad G = u_1 g(\tau,\sigma) - 3u_1^2\sigma^2 f(\tau,\sigma),$$

where $\tau = t + u$. Invoking the operator $qt\partial_t + (1+q)x\partial_x + qu\partial_u$ as a symmetry, we obtain the system

$$\begin{aligned} qtF_t + quF_u &= u_1 F_{u_1} + (q+2)u_2 F_{u_2} + (2q+3)F \\ qtG_t + quG_u &= u_1 G_{u_1} + (q+2)u_2 G_{u_2}, \end{aligned}$$

which then gives

$$\begin{aligned} q\tau f_\tau &= (q-1)\sigma f_\sigma + 2qf \\ q\tau g_\tau &= (q-1)\sigma g_\sigma + g. \end{aligned}$$

We then obtain:

$$F = u_1^{-3}(t+u)^2\phi(\omega), \quad G = u_1(t+u)^{1/q}\psi(\omega) - 3u_1^2\sigma^2(t+u)^2\phi(\omega), \quad \omega = \sigma\tau^{(q-1)/q}$$

for $q \neq 0$, and

$$F = u_1^{-3}\phi(\tau), \quad G = u_1\sigma\psi(\tau) - 3u_1^2\sigma^2\phi(\tau)$$

if $q = 0$.

$\underline{\langle \partial_x, u\partial_x, \partial_t - \partial_u, \alpha\partial_t + x\partial_x \rangle}$. In this case we have, as above,

$$F = u_1^{-3} f(\tau,\sigma), \quad G = u_1 g(\tau,\sigma) - 3u_1^2\sigma^2 f(\tau,\sigma),$$

where $\tau = t + u$. Invoking the operator $\alpha\partial_t + x\partial_x$ as a symmetry, we obtain the system

$$\begin{aligned} \alpha F_t &= u_1 F_{u_1} + 2u_2 F_{u_2} + (3-\alpha)F \\ \alpha G_t &= u_1 G_{u_1} + 2u_2 G_{u_2}, \end{aligned}$$

which then gives

$$\alpha f_\tau + \sigma f_\sigma = 0, \quad \alpha g_\tau + \sigma g_\sigma = g.$$

On integrating we find

$$f(\tau,\sigma) = \phi(\omega), \quad g(\tau,\sigma) = \sigma\psi(\omega), \quad \omega = \sigma e^{-\tau/\alpha}, \quad \alpha \neq 0$$

and

$$f = \phi(\tau), \quad g = \sigma\psi(\tau), \quad \alpha = 0.$$

Finally, if $\alpha \neq 0$

$$F = u_1^{-3}\phi(\omega), \quad G = u_1^{-2}u_2\psi(\omega) - 3u_1^{-4}u_2^2\phi(\omega),$$

and if $\alpha = 0$

$$F = u_1^{-3}\phi(\tau), \quad G = u_1^{-2}u_2\psi(\tau) - 3u_1^{-4}u_2^2\phi(\tau).$$

$\underline{\langle \partial_x, \partial_t, t\partial_x - \partial_u, t\partial_t + (1+q)x\partial_x + qu\partial_u\rangle}$. In this case we have $F_t = F_x = G_t = G_x = 0$ from the first two operators and then the system

$$F_u = 0, \quad G_u = u_1$$

from $t\partial_x - \partial_u$. These two equations give

$$F = F(u_1, u_2), \quad G = uu_1 + g(u_1, u_2).$$

Then the operator $t\partial_t + (1+q)x\partial_x + qu\partial_u$ gives rise to the system

$$\begin{aligned} u_1F_{u_1} + (q+2)u_2F_{u_2} + (3q+2)F &= 0 \\ u_1g_{u_1} + (q+2)u_2g_{u_2} + (q-1)g &= 0 \end{aligned}$$

and these integrate to give

$$F = u_1^{-(3q+2)} f\left(\frac{u_2}{u_1^{q+2}}\right), \quad G = uu_1 + u_1^{1-q} g\left(\frac{u_2}{u_1^{q+2}}\right).$$

The evolution equation is then

$$u_t = u_1^{-(3q+2)} f\left(\frac{u_2}{u_1^{q+2}}\right) u_3 + uu_1 + u_1^{1-q} g\left(\frac{u_2}{u_1^{q+2}}\right).$$

The KdV-equation corresponds to the case $f = 1,\ g = 0,\ q = -2/3$.
$\underline{\langle \partial_x, \partial_t, t\partial_x - \partial_u, t\partial_t + x\partial_x + \kappa\partial_u\rangle}$. As before, the first three operators give us

$$F = F(u_1, u_2), \quad G = uu_1 + g(u_1, u_2).$$

Then the operator $t\partial_t + x\partial_x + \kappa\partial_u$ gives rise to the system

$$\begin{aligned} u_1F_{u_1} + 2u_2F_{u_2} + 2F &= 0 \\ u_1G_{u_1} + 2u_2G_{u_2} + G &= \kappa G. \end{aligned}$$

With F, G as above, these equations yield

$$F = u_1^{-2} f\left(\frac{u_2}{u_1^2}\right), \quad G = uu_1 + \kappa u_1 \ln|u_1| + u_1 g\left(\frac{u_2}{u_1^2}\right).$$

$\underline{\langle \partial_x, \partial_t, (t+u)\partial_x, t\partial_t + (1+q)x\partial_x + u\partial_u \rangle}$. The first three operators give us $F_t = F_x = \underline{G_t = G_x = 0 \text{ and the operator } (t+u)\partial_x}$ gives the system

$$u_1 F_{u_1} + 3u_2 F_{u_2} + 3F = 0$$
$$u_1^2 G_{u_1} + 3u_1 u_2 G_{u_2} = u_1 + u_1 G - 3u_2^2 F.$$

These integrate to give

$$F = u_1^{-3} f(u, \omega), \quad G = u_1 g(u, \omega) - 3u_1^2 \omega^2 f(u, \omega) - 1,$$

where $\omega = u_1^{-3} u_2$. Then the operator $t\partial_t + (1+q)x\partial_x + u\partial_u$ gives the system

$$uf_u + (q-1)\omega f_\omega = 2f$$
$$ug_u + (q-1)\omega g_\omega = qg,$$

which leads to

$$f = u^2 \phi(\rho), \quad g = u^q \psi(\rho), \quad \rho = u^{1-q}\omega = u^{1-q} u_1^{-3} u_2$$

and

$$F = u_1^{-3} u^2 \phi(\rho), \quad G = u_1 u^q \psi(\rho) - 3(u_1^{-1} u_2)^4 \phi(\rho) - 1.$$

We conclude by presenting the remaining algebras and the corresponding invariant equations, whenever possible, in tables.

$$A_{4.7}^1 = A_{3.3}^1 \niplus \langle t\partial_t + (x-t)\partial_x + (2u - \frac{1}{2}t^2)\partial_u \rangle;$$
$$F = \tilde{F}(u_{xx}) u_x^2, \quad G = x + u_x(\ln|u_x| + \tilde{G}(u_{xx})),$$

$$A_{4.7}^2 = A_{3.3}^2 \niplus \langle -\partial_t + x\partial_x + 2u\partial_u \rangle;$$
$$F = \tilde{F}(u_{xx}) e^{-3t}, \quad G = -\frac{1}{2}u_x^2 + \tilde{G}(u_{xx}) e^{-2t},$$

$$A_{4.9}^1 = A_{3.3}^2 \niplus \langle -(1+t^2)\partial_t + (q-t)x\partial_x + (2qu - \frac{1}{2}x^2)\partial_u \rangle \quad (q > 0);$$
$$F = \tilde{F}(\omega)(1+t^2)^{1/2} \exp(-3q \arctan t),$$
$$G = -\frac{1}{2}u_x^2 + \tilde{G}(\omega)(1+t^2)^{-1} \exp(-2q \arctan t), \quad \omega = t - (1+t^2) u_{xx}.$$

$$A_{4.10}^1 = A_{3.5}^3 \niplus \langle 2kt\partial_t + u\partial_x - x\partial_u \rangle, \quad k \geq 0;$$

$$F = t^{1/2} \exp(3k \arctan u_x)(1+u_x^2)^{-3/2} \tilde{F}(\sigma),$$
$$G = t^{-1/2} \exp(k \arctan u_x)(1+u_x^2)^{1/2}[-3\sigma^2 u_x \tilde{F}(\sigma) + \tilde{G}(\sigma)],$$
$$\sigma = t^{1/2} u_{xx} \exp(k \arctan u_x)(1+u_x^2)^{-3/2}.$$

We sum up the above results as a theorem.

Table 4: Equations invariant under four-dimensional nondecomposable algebras

Algebra	F	G
$A^1_{4.2}$	$\tilde{F}(\omega)\exp(3-q)u_x$	$\exp(1-q)u_x\tilde{G}(\omega), \quad q\neq 0,1,\ \omega=u_{xx}\exp u_x$
$A^2_{4.2}$	$\tilde{F}(\omega)\lvert u_x\rvert^{\frac{2}{q-1}}$	$u_x[(1-q)^{-1}\ln\lvert u_x\rvert+\tilde{G}(\omega)], \quad q\neq 0,1,\ \omega=u_x^{\frac{2-q}{q-1}}u_{xx}$
$A^1_{4.3}$	$\tilde{F}(u_{xx})\exp(-u_x)$	$\tilde{G}(u_{xx})\exp(-u_x)$
$A^2_{4.3}$	$\tilde{F}(\omega)$	$u_x[\tilde{G}(\omega)-u_x\ln\lvert u_x\rvert], \quad \omega=u_x^{-1}u_{xx}$
$A^1_{4.4}$	$\tilde{F}(\omega)\exp 2u_x$	$-\frac{1}{2}u_x^2+\tilde{G}(\omega), \quad \omega=u_{xx}\exp u_x$
$A^1_{4.5}$	$u_x^{(1-3p)/(p-q)}\tilde{F}(\omega)$	$u_x^{(1-p)/(p-q)}\tilde{G}(\omega), \quad \omega=u_x^{\frac{2p-q}{q-p}}u_{xx}$
$A^1_{4.6}$	$\tilde{F}(\omega)(1+u_x^2)^{-3/2}\times$ $\exp[(q-3p)\arctan u_x]$	$-3u_x(1+u_x^2)^{-5/2}u_{xx}^2\exp[(q-3p)\arctan u_x]\tilde{F}+$ $+\exp[(q-p)\arctan u_x](1+u_x^2)^{1/2}\tilde{G}(\omega),$ $\omega=(1+u_x^2)^{3/2}\exp(p\arctan u_x)u_{xx}^{-1}$

Theorem 6.4 *There exist thirty inequivalent four-dimensional symmetry algebras admitted by equation (2.6). The explicit forms of those algebras as well as the associated invariant equations are given above.*

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